THE UNIVERSITY OF ALBERTA

SINGULAR INTEGRALS WHOSE KERNELS INVOLVE CERTAIN STURM-LIOUVILLE FUNCTIONS

by

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR

THE DEGREE OF

DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

EDMONTON, ALBERTA

1966

UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "Singular Integrals whose Kernels Involve Certain Sturm-Liouville Functions" submitted by Martin Eugene Muldoon in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

The principal object of study here is the behaviour, as $\nu \to \infty$, of the integral

$$(1) \qquad \int_a^\infty f(t)\, w(\nu, t-x)\, dt,$$

where $-\infty < a < x < \infty$, and for each $\nu > 0$, $w(\nu,t)$ is a suitably normalized solution, vanishing at ∞, of the differential equation

$$(2) \qquad d^2w/dt^2 = [\nu^2 t + q(t)]w.$$

We assume that $q(t)$ is continuous for $a - x \leq t < \infty$, and that $\int_{a-x}^\infty |t^{-1/2}\, q(t)|\, dt$ exists.

The following is a simplified form of one of the main results. Let $f(t)$ be defined on $a \leq t < \infty$, where $-\infty < a < 0$, and suppose that (i) $f \in L(a,\infty)$, (ii) $f(0+)$ exists, and (iii) $f \in BV[a,0]$; then

$$(3) \qquad \lim_{\nu\to\infty} \int_a^\infty f(t)\, w(\nu,t)\, dt = \tfrac{2}{3}f(0-) + \tfrac{1}{3}f(0+).$$

We consider applications of (3) to integrals involving some well-known special functions which satisfy, on making appropriate changes of variable, differential equations of type (2). The special case $q(t) \equiv 0$ corresponds to $w(\nu,t) = \nu^{2/3}\, Ai(\nu^{2/3}t)$, with the usual notation for the Airy function.

We show that if $w(\nu,t)$ is given by this expression involving the Airy Function, hypothesis (iii) in the above theorem cannot be weakened to : (iii') $f(0-)$ exists. This therefore raises the question of the summability analogue of (3). It turns out that the result corresponding to (3) holds under hypotheses (i), (ii), and (iii'), provided

$w(\nu,t)$ in (3) is replaced by its Cesàro (C,k) mean, $k > 1/2$, and provided the integrals involved in the (C,k) process exist. This result is not valid for $k = 1/2$, at least in the case $w(\nu,t) = \nu^{2/3}\,Ai(\nu^{2/3}t)$.

We show that when f belongs to a suitable class of functions and $\rho > 0$, the Gibbs phenomenon is exhibited at the point $x = 0$ by

$$\int_a^\infty f(t)\ \nu^{\rho'} Ai[\nu^{\rho'}(t-x)]\ dt.$$

We consider some aspects of the Gibbs phenomenon as exhibited by this expression, in the case of (C,k) summability, $k > 0$.

ACKNOWLEDGEMENTS

I would like to thank Professor Lee Lorch for introducing me to this area of research, supervising the work, and suggesting a large number of problems, only some of which are considered here. Dr. E. L. Whitney helped with the supervision of the work during 1964-65; I want to thank him for this and for his continued interest. I am indebted to Mr. David Klarner, who has helped by proofreading, and in other ways, to expedite the appearance of the thesis in final form.

I am grateful for the assistance of the Department of Computing Science, University of Alberta, in performing certain calculations which led to some of the results and conjectures of Chapter 6.

The financial support of the University of Alberta and the National Research Council of Canada is also gratefully acknowledged.

CONTENTS

INTRODUCTION

The name "singular integral" has been traditionally applied to integrals of the form:

$$(0.1) \qquad \int_a^b f(t)\Phi_\nu(t,x)\,dt, \quad a<x<b,$$

where the problem of interest is the existence of

$$(0.2) \qquad \lim_{\nu\to\infty}\int_a^b f(t)\Phi_\nu(t,x)\,dt$$

and its relation to $f(x)$, under various assumptions on f and Φ. The function of $\Phi_\nu(t,x)$ has generally been called the "kernel" of the singular integral. The sense in which we will use these terms is given in Definition 1.1.

Integrals of the form (0.1) were first considered in connection with Fourier series. Dirichlet (in 1829) introduced the expression

$$(0.3) \qquad S_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin[(n+\frac{1}{2})(t-x)]}{\sin\frac{1}{2}(t-x)} f(t)\,dt$$

for the (2n+1)-st. partial sum of the Fourier Series of $f(x)$ and this so-called "Dirichlet singular integral" became the basis for many later investigations. A discussion of Dirichlet's work and of subsequent developments is given in Chapter 8 of E. W. Hobson's book ([16], vol. 2).

L. Fejér ([11], 1904) introduced the expression

$$(0.4) \qquad \sigma_n(x) = \frac{1}{2n\pi}\int_{-\pi}^{\pi}\left[\frac{\sin\frac{1}{2}n(t-x)}{\sin\frac{1}{2}(t-x)}\right]^2 f(t)\,dt,$$

where $\sigma_n(x)$ is given by

$$(0.5) \qquad \sigma_n(x) = \frac{1}{n+1}[S_o(x)+S_1(x)+\ldots+S_n(x)].$$

Fejér proved that $\sigma_n(x)$ converges to $f(x)$ at all points of continuity and to $\frac{1}{2}f(x+) + \frac{1}{2}f(x-)$ at points of simple discontinuity of any Riemann integrable function f. Later, H. Lebesgue showed that $\sigma_n(x)$ converges to $f(x)$ almost everywhere on $[-\pi,\pi]$. In 1876, du Bois Reymond had shown that $S_n(x)$ need not converge to $f(x)$ at all points of continuity. Thus it appears that the Fejér kernel in (10.4) is much more effective than the Dirichlet kernel in securing the convergence of the corresponding singular integral.

Singular integrals also arose in connection with some of the proofs of the Weierstrass approximation theorem. First proved by Weierstrass in 1885, the theorem may be stated as follows:

If $f(x)$ is continuous in $[0,1]$, there exists a sequence of polynomials $\{p_n(x)\}$ converging uniformly to $f(x)$ in $[0,1]$.

Subsequent proofs, using a variety of methods, were given by H. Lebesgue (1908), E. Landau (1908), C. J. de La Vallée Poussin (1908) and S. Bernstein (1912).

Landau's proof [17] uses the singular integral

$$(0.6) \qquad L_n(x) = \sqrt{\frac{n}{\pi}}\int_0^1 [1-(t-x)^2]^n\, f(t)\, dt.$$

Since $L_n(x)$ is a polynomial in x for each n, Landau was able

to prove the theorem by showing that $L_n(x) \to f(x)$ uniformly in x, $0 \leq x \leq 1$, as $n \to \infty$.

La Vallée Poussin [18] proves a trigonometrical equivalent of the Weierstrass theorem by showing that

$$(0.7) \qquad V_n(x) = \frac{\sqrt{n}}{2\sqrt{\pi}} \int_{-\pi}^{\pi} \cos^{2n} \frac{(t-x)}{2}\, f(t)\, dt$$

approaches $f(x)$, uniformly in x, as $n \to \infty$.

Singular Integrals also arise naturally in convergence problems of general orthogonal series. An account of some results appropriate to these applications is contained in Chapter 4 of G. Alexits' book [2]. A description of how singular integrals arise in some of these cases and also in the case of Fourier series is given in a 1949 article by Fejér [12].

All of the results described above refer to special kernels of the form $\Phi_\nu(t-x)$ and are concerned with conditions on f which will validate a relation of the type:

$$(0.8) \qquad \lim_{\nu \to \infty} \int_a^b f(t)\, \Phi_\nu(t-x)\, dt = f(x).$$

Adopting a slightly different point of view, H. Lebesgue in 1909 gave a systematic theory [19] in which necessary and sufficient conditions on Φ were found which would imply a result such as (0.8) for all functions f belonging to a particular class, e.g. the class of continuous or of summable functions or of functions of bounded variation on an interval $[a,b]$. Lebesgue showed, *inter alia*, that in order for (0.8)

to hold for all f continuous at a point x, it is necessary that

$$(0.9) \qquad \int_a^b |\Phi_\nu(t-x)| \, dt < K,$$

where K is independent of ν. This serves to clarify one distinction between the Fejér and Dirichlet kernels since the condition (0.8) is satisfied for the former but not for the latter.

Subsequent work on general singular integrals in the one-dimensional case consists largely of improvements and modifications of Lebesgue's results. E. W. Hobson ([16], vol. 2, ch. 7) finds necessary and sufficient conditions for the convergence of $\int_a^b f(t) \, \Phi_\nu(t,x) \, dt$ where $\Phi_\nu(t,x)$ need not have the special form $\Phi_\nu(t-x)$. He pays special attention to the uniformity in x of the convergence. During the 1930's, P. Romanovski [28], D. K. Faddeyev [10], and I. P. Natanson contributed to the study (initiated by Lebesgue [19]) of the convergence of singular integrals at points where f satifies:

$$(0.10) \qquad \lim_{h\to 0} h^{-1} \int_0^h [f(t) - f(x)] \, dt = 0$$

or

$$(0.11) \qquad \lim_{h\to 0} h^{-1} \int_0^h | f(t) - f(x)| \, dt = 0.$$

Such a study is important in that a summable function satisfies these conditions almost everywhere. An account of these

results is given in Natanson's book ([24], Chapter 10).

In recent years a number of authors, including A. P. Calderón and A. Zygmund have used the designation "singular integrals" for certain generalizations in n-dimensional space of the classical Hilbert transform:

$$\tilde{f}(x) = \lim_{\varepsilon\to 0} \frac{1}{\pi} \int_{|t-x| \geq \varepsilon} \frac{f(t)}{x-t}\, dt \tag{0.12}$$

Calderón gives an account of this theory in a recent survey article [3]. We shall not use the term "singular integral" in the sense of these authors.

The topics discussed in the present study arose from some results of L. Lorch and P. Szego ([20] and [21]). One of their results was the following (in which J_ν denotes the Bessel function of the first kind):

<u>Theorem 0.1</u> ([20], Theorem 1). Let $f(t)$ be defined on $[0,A]$, $1<A<\infty$ and suppose that:

(i) $t^\lambda f(t) \varepsilon L[0,A]$ for some real λ; (ii) $f(1-)$ exists; (iii) $f \varepsilon BV [1,A]$. Then

$$\lim_{\nu\to\infty} \int_0^A f(t)\, \nu J_\nu(\nu t)\, dt = 1/3\, f(1-) + 2/3 f(1+). \tag{0.13}$$

Lorch and Szego remark that this result is unusual in that $f(1-)$ and $f(1+)$ enter with unequal weights and that stronger hypotheses are required for f to the right of t=1 than to the left, these peculiarities being due to the change in behaviour

of $J_\nu(\nu t)$ from nonoscillatory to oscillatory at the point $t=1$; the kernel behaves somewhat like the Fejér kernel for $t<1$ and like the Dirichlet kernel for $t>1$.

Now this change in behaviour is shared by many well-known special functions which satisfy second order linear differential equations. From the point of view of differential equations, one of the simplest such functions is the Airy function $Ai(t)$ which is oscillatory for $t<0$, positive and monotonically decreasing for $t\geq 0$ and satisfies:

$$Ai''(t) = t\,Ai(t). \tag{0.14}$$

In Chapter 2 we establish some results analogous to Theorem 0.1 for the integral

$$\int_a^\infty f(t)\,\nu\,Ai(\nu t)\,dt, \quad -\infty < a < 0. \tag{0.15}$$

It is known that, under certain conditions on $q(t)$, the differential equation

$$\frac{d^2w}{dt^2} = [\nu^2 t + q(t)]w \tag{0.16}$$

possesses a solution $w(\nu,t)$ which, for large ν, is approximately $\nu^{2/3}Ai(\nu^{2/3}t)$, so it seems reasonable that the integral

$$\int_a^\infty f(t)\,w(\nu,t)\,dt, \ -\infty < a < 0, \tag{0.17}$$

may share some of the properties of (0.15). Chapter 3 of the present work is devoted to proving a precise form of

this statement.

Many of the important special functions of analysis satisfy (on making suitable changes of variable) differential equations of type (0.16). In Chapter 4, we use the results of Chapter 3 to rederive Theorem 0.1 of Lorch and Szego and to get Theorems similar to Theorem 0.1 for singular integrals involving the Whittaker function $W_{k,m}(x)$, the parabolic cylinder function $D_\nu(x)$, and various specializations of these.

Lorch and Szego show that the hypothesis "$f \in BV[1,A]$" of Theorem 0.1 may not be replaced by "$f(1+)$ exists" in the case of ordinary convergence in (0.13) or even in the case when the limit relation in (0.13) is interpreted in the sense of Cesàro (C,k) summability, $0 < k \leq 1/2$; they show, however, that this replacement may be made in the case of (C,k) summability, $k > 1/2$. They also discuss certain aspects of the Gibbs phenomenon as exhibited by the singular integral in (0.13). Chapters 5 and 6 of the present work are devoted to a study of questions regarding Cesàro summability and the Gibbs phenomenon in the case of the singular integral

$$\int_a^\infty f(t)\, w(\nu,t)\, dt, \quad -\infty < a < 0,$$

in special cases where $w(\nu,t)$ involves the Airy function $Ai(t)$. The results obtained are very similar to those of Lorch and Szego and they go further in some directions than the corresponding results of these authors. The definitions of Cesàro summability and of the Gibbs phenomenon appropriate to

the present situation are given in the introductions to Chapters 5 and 6.

Notation

All of the quantities discussed throughout this thesis are assumed to be real. We shall use the notations [a,b], [a,b) and (a,b) to denote the intervals $a \leq x \leq b$, $a \leq x < b$ and $a < x < b$, respectively. We say that $f \in L(a,b)$ if

$$\int_a^b f(t)\,dt$$ exists in the Lebesgue sense and has a finite value. All of the functions which we consider are assumed to be measurable.

CHAPTER I

GENERAL THEOREMS ON SINGULAR INTEGRALS

We now state four theorems (due principally to Lebesgue) on the convergence of singular integrals. Some of these results will be used in later chapters.

Theorem 1.1 Let $\Psi_\nu(t,x)$ be a measurable function of t on $[a,b]$, $-\infty < a < b < \infty$, for each real $\nu \geq \nu_0$ and for each x in a set E of real numbers. Let

(1.1) $|\Psi_\nu(t,x)| \leq K$ for all ν, t, and x considered;

(1.2) $\lim\limits_{\nu\to\infty}\int_\alpha^\beta \Psi_\nu(t,x)\,dt = 0$, uniformly in x, $x \in E$, for each subinterval $[\alpha,\beta]$ of $[a,b]$;

(1.3) $f \in L\,[a,b]$.

Then

(1.4) $\lim\limits_{\nu\to\infty}\int_a^b f(t)\Psi_\nu(t,x)\,dt = 0$, uniformly in x, $x \in E$.

The result (1.4) holds in case $b = \infty$, provided (1.3) holds for each b, $a < b < \infty$,

$$\lim_{b\to\infty}\int_a^b f(t)\,dt$$

exists as a finite limit, $\Psi_\nu(t,x)$ satisfies the condition (1.1) on the interval $[a,\infty)$ the condition (1.2) on each finite subinterval $[\alpha,\beta]$ of $[a,\infty)$, and in addition:

(1.5) $V_a^\infty\,[\Psi_\nu(t,x)] \leq L$, where L is independent of ν and x.

Remarks. The theorem is due in essence to Lebesgue, ([19],

p. 52) though he does not include uniformity in x, in either hypotheses or conclusion. The proof of Theorem 1.1 is contained in Hobson([16], vol. 2, pp. 422-430).

The terms "kernel" and "singular integral" are not defined in the same way by all authors. We follow closely the terminology used by Romanovski in [27].

Definition 1.1 Let $\Phi(\nu,t)$ be defined as a measurable function of t on $a \leq t \leq b$ $(\leq \infty)$ for each $\nu \geq \nu_0$ and let

$$(1.6) \qquad \lim_{\nu\to\infty} \int_a^\beta \Phi(\nu,t)\, dt = \ell,$$

for each β satisfying $a \leq \beta \leq b$, and uniformly in β, $a + \epsilon \leq \beta \leq b$, for each $\epsilon > 0$. Then $\Phi(\nu,t)$ will be called a kernel on the interval $[a,b]$, with singular point a, and integral value ℓ. An integral of the form

$$\int_x^b f(t)\; \Phi(\nu,t-x)\, dt$$

will be called a singular integral.

Note. We will find it convenient to use the same expressions in the case when the singular point is an interior point of the interval considered and the definitions are modified in an obvious way.

We now state some theorems on the convergence of singular integrals.

Theorem 1.2

(i) Let $\Phi(\nu,t)$ be a kernel on $[0, b+\delta]$, where $0 < \delta < b < \infty$, with singular point 0 and integral value ℓ. Let $\Phi(\nu,t)$ satisfy also the following conditions:

(1.7) $$|\Phi(\nu,t)| \leq K(\delta_1),\ \delta_1 \leq t \leq b+\delta,\ \nu \geq \nu_0,$$

for each δ_1 satisfying $0 < \delta_1 \leq b + \delta$;

(1.8) $$\int_0^{b+\delta} |\Phi(\nu,t)|\,dt < K \text{ for all } \nu \geq \nu_0;$$

suppose that f is defined on $[-\delta, b]$ and let:

(1.9) $$f \in L[-\delta,b];$$

(1.10) $$f \in C[-\delta,\delta];$$

(1.11) $$\lim_{h\to 0+} f(\delta+h) = f(\delta).$$

Then

(1.12) $$\lim_{\nu\to\infty} \int_x^b f(t)\,\Phi(\nu,t-x)\,dt = \ell\, f(x),$$

uniformly in x, $-\delta \leq x \leq \delta$.

(ii) If $\int_0^b |\Phi(\nu,t)|\,dt$ is an unbounded function of ν, as $\nu \to \infty$, there exists a function f, continuous on $[0,b]$, for which $\int_0^b f(t)\,\Phi(\nu,t)\,dt$ is an unbounded function of ν.

Remarks. Lebesgue ([19], p. 70) proves the result (i) apart from the uniformity assertion. Hobson ([16], vol. 2, pp. 447-448) proves a result which implies (i). Note that in (i) the case $\delta = 0$ is not excluded. (ii) was also proved by

Lebesgue. We present the proof of (i) as an example of the way in which convergence theorems for singular integrals are proven.

<u>Proof of (i)</u>. Let $\epsilon > 0$ be given. By (1.10) and (1.11) there exists δ_1 such that, for each $x \varepsilon [-\delta,\delta]$, $x < t < x+\delta_1$ implies $|f(t) - f(x)| < \epsilon$. Then

$$\text{(1.13)} \qquad \int_x^b f(t)\ \Phi(\nu,t-x)\ dt = f(x)\int_x^{x+\delta_1} \Phi(\nu,t-x)\ dt$$
$$+\int_x^{x+\delta_1} [f(t) - f(x)]\ \Phi(\nu,t-x)\ dt$$
$$+\int_{x+\delta_1}^b [f(t)]\ \Phi(\nu,t-x)\ dt.$$

Now the first term on the right of (1.13) approaches f(x) uniformly in x, $-\delta \leq x \leq \delta$, on account of the fact that $\Phi_\nu(t)$ is a kernel. (1.8) shows that the second term is bounded by εK, for all $\nu \geq \nu_0$ and $-\delta \leq x \leq \delta$; to deal with the third term we use Theorem 1.1 in which we place

$$\text{(1.14)} \qquad \Psi_\nu(t,x) = \begin{cases} 0 & , -\delta + \delta_1 < t < x + \delta_1, \\ \Phi_\nu(t-x), & x + \delta_1 < t < b. \end{cases}$$

Now clearly, on account of (1.7), $|\Psi_\nu(t,x)| \leq K(\delta_1)$ for all ν,t and x concerned and f(t) is integrable over $[-\delta+\delta_1, b]$ by (1.17). $\Psi_\nu(t,x)$ has the property (1.2) because $\Phi_\nu(t)$ has the kernel property (1.6). Thus Theorem 1.1 shows that

$$\lim_{\nu\to\infty} \int_{-\delta+\delta_1}^b f(t)\ \Psi_\nu(t,x)\ dt = 0, \text{ uniformly in } x,\ -\delta \leq x \leq \delta.$$

Hence:

$$\lim_{\nu\to\infty} \int_{x+\delta_1}^{b} f(t)\ \phi_\nu(t-x)\ dt = 0, \tag{1.15}$$

uniformly in x, $-\delta \leq x \leq \delta$.

Thus we see that the left-hand side of (1.13) tends to f(x) uniformly in x and so the proof of (i) is complete.

<u>Remarks on (ii)</u>. Lebesgue proved this result by showing how a function f with the desired property may be constructed. It is now known that the existence of such a function is a consequence of the principle of uniform boundedness. This principle, which had its genesis, in part, in this very result of Lebesgue, may be stated as follows:

> If $\{\Phi_n\}$ be a sequence of bounded linear functionals on a Banach space B and if for each $f \in B$ the sequence $\{|\Phi_n f|\}$ is bounded, then the sequence of norms $\{\|\Phi_n\|\}$ is bounded.

To apply this in the present case we consider the Banach space C[0,b] of continuous functions on [0,b], with the sup norm and let

$$\Phi_\nu f = \int_0^b f(t)\quad \phi(\nu,t)\ dt. \tag{1.16}$$

We can show that for each ν, Φ_ν is a bounded linear functional on C[0,b] and

$$\|\Phi_\nu\| = \int_0^b |\phi(\nu,t)|\ dt \tag{1.17}$$

If there exists no f such that $|\Phi_\nu f|$ is unbounded in ν, the principle of uniform boundedness shows that $\{\int_0^b |\phi(\nu,t)|\,dt\}$ is

a bounded function of ν, a contradiction to the hypothesis of (ii). Hence the conclusion of (ii) holds.

We state two more theorems on the convergence of singular integrals.

Theorem 1.3. (Romanovsky [27], Natanson [24]). Let $\Phi(\nu,t)$ be a kernel on $(0,b)$, $0 < b < \infty$, with singular point 0 and integral value ℓ, and suppose that for each $\nu > \nu_o$, $\Phi(\nu,t)$ is a positive non-increasing function of t, $0 < t < b$. Let

(1.18) $$f \varepsilon L\,[0,b], \text{ and}$$

(1.19) $$\lim_{h\to 0+} h^{-1} \int_0^h f(t)\,dt = c.$$

Then

(1.20) $$\lim_{\nu\to\infty} \int_0^b f(t)\,\Phi(\nu,t)\,dt = c\ell.$$

Remark. The conditions imposed on $\Phi(\nu,t)$ are stronger than the corresponding conditions in Theorem 1.2 (case $\delta=0$). For this reason, the condition (1.19) on f is weaker than the corresponding condition (1.11). Theorem 1.4 below goes in the opposite direction—stronger conditions on f, weaker on Φ.

Theorem 1.4 (Lebesgue [19], p. 70). Let $\Phi(\nu,t)$ be a kernel on $(0,b)$, $0 < b < \infty$, with singular point 0 and integral value ℓ, and let:

(1.21) $$\left|\int_\lambda^\mu \Phi(\nu,t)\,dt\right| < K,$$

for all $\nu \geq \nu_o$, and all λ, μ satisfying $0 \leq \lambda\ \mu \leq b$. Let

(1.22) $$f \,\varepsilon\, BV[0,b].$$

Then

$$(1.23) \qquad \lim_{\nu\to\infty}\int_0^b f(t)\ \Phi(\nu,t)\ dt = \ell f(0+).$$

Note: The theorems of this chapter are not always quoted in the form in which they appear in the cited references. In particular, many of the authors quoted state the theorems only for the case when ν approaches infinity through integer values. It is clear, however, that we may take ν to be a continuous variable in all the cases considered.

CHAPTER II

A KERNEL INVOLVING AN AIRY FUNCTION

The Airy function Ai(x) may be defined by:

$$(2.1)\qquad Ai(x) = \pi^{-1}\int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) dt, \ -\infty < x < \infty.$$

It can be shown (see e.g., Watson [34], p. 170) that

$$(2.2)\qquad \begin{aligned} Ai(x) &= \pi^{-1}(x/3)^{\frac{1}{2}} K_{1/3}\left(\frac{2}{3}x^{3/2}\right), \ x \geqslant 0, \\ Ai(-x) &= \frac{1}{3}x^{\frac{1}{2}}[J_{1/3}(\tfrac{2}{3}x^{3/2}) + J_{-1/3}(\tfrac{2}{3}x^{3/2})], \ x \geqslant 0, \end{aligned}$$

with the usual notation for Bessel functions. Ai(x) satisfies the differential equation:

$$(2.3)\qquad Ai''(x) = x\,Ai(x).$$

The known asymptotic formulas for Bessel functions (see, e.g., [34], Chapter 7), together with (2.2), give:

$$(2.4)\qquad Ai(x) = \frac{1}{2}\pi^{-\frac{1}{2}}x^{-\frac{1}{4}}\exp(-2/3\,x^{3/2})[1 + R_1(x)], \ 0 < x < \infty,$$

and

$$(2.5)\qquad Ai(-x) = \pi^{-\frac{1}{2}}x^{-\frac{1}{4}}[\cos(2/3x^{3/2} - \pi/4) + R_2(x)], \ 0 < x < \infty,$$

where

$$R_1(x),\ R_2(x) < C\,x^{-3/2},\ 0 < x < \infty, \text{ for some constant } C.$$

Using (2.2), together with [34], p. 388,(8), we get

$$(2.6)\qquad \int_0^\infty Ai(x)\,dx = 1/3,$$

while (2.3), in conjunction with [34], p. 391, (1) gives

(2.7) $$\int_0^\infty Ai(-x)\, dx = 2/3.$$

A summary of results connected with the Airy function is given in [1], §10.4.

Lorch and Szego ([21], p. 210) use a result of E. Makai [22]* to give a simple proof that for $x>o$, $Ai(-x)$ is an oscillating function whose successive arches bound decreasing areas. (The first arch is from $x = 0$ to $x = a$, the first positive root of $Ai(-x)$. Hence, by an "alternating series argument" there exists a constant M such that

(2.8) $$\left|\int_0^t Ai(-x)\, dx\right| < M, \text{ for all } t \geq 0.$$

<u>Lemma 2.1</u>

$Ai(x) \exp(2/3\, x^{3/2})$ is a positive decreasing function of x for $x \geq 0$.

<u>Proof</u>

In view of (2.2) it will be sufficient to show that if

$$F(x) = e^x x^{1/3} K_{1/3}(x)$$

then $F(x)$ is a positive decreasing function of x for $x > 0$.

We have

(2.9) $$F'(x) = e^x x^{1/3} K_{1/3}(x) - e^x x^{1/3} K_{-2/3}(x),$$

([34], p. 79 (5)),

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*The same conclusion might have been drawn from a result of P. Hartman and A. Wintner, Amer. J. Math, vol. 70 (1948), pp. 529-539; cf. also [15], pp. 510-513.

$$= e^{x} x^{1/3} \int_0^\infty e^{-x \cosh t}[\cosh\tfrac{t}{3} - \cosh\tfrac{2t}{3}]\, dt,$$

([34], p. 181, (5)),

< 0, for $x>0$.

Hence $F(x)$ is decreasing. The integral representation for $K_{1/3}(x)$ just quoted also shows that $F(x) > 0$ for $x \geq 0$. This completes the proof of Lemma 2.1.

Corollary $Ai(x) \exp(\lambda x^{3/2})$ is a positive decreasing function of x $(x \geq 0)$ for each real λ satisfying $-\infty < \lambda \leq 2/3$.

Lemma 2.2 (due to Natanson, and proved in [24], p.16)

Let $f,g \in L[0,\eta]$, $0 < \eta < \infty$, and let

(2.10) $0 \leq g \in \downarrow [0,\eta]$;

(2.11) $|h^{-1}\int_0^h f(t)\, dt| < \epsilon$, $0 < h \leq \eta$;

Then $\int_0^\eta f(t)\, g(t)\, dt$ exists, and

(2.12) $$|\int_0^\eta f(t)g(t)\, dt| \leq \epsilon \int_0^\eta g(t)\, dt.$$

We now prove two theorems on singular integrals involving $Ai(t)$.

Theorem 2.1

Let $f(t)$ be defined on $[-\delta,\infty)$, where $\delta \geq 0$, and suppose that $f \in L[-\delta,b]$, for each b satisfying $-\delta < b < \infty$, and

(2.13) $\lim_{b\to\infty} \int_{-\delta}^{b} f(t) \exp[-\lambda(t+\delta)^{3/2}]\,dt$ exists and is finite for some real λ;

(2.14) $$\lim_{h\to 0+} h^{-1}\int_x^{x+h} [f(t) - f(x)]\, dt = 0,$$

uniformly in x, $-\delta \leq x \leq \delta$.

Then

(2.15) $$\lim_{\nu\to\infty} \int_x^\infty f(t)\ \nu\ \mathrm{Ai}[\nu(t-x)]\ dt = (1/3)\ f(x),$$

uniformly in x, $-\delta \leq x \leq \delta$.

Remark 1. Notice that the case $\delta=0$ is included.

Remark 2. It is clear from (2.6) that $\nu\mathrm{Ai}(\nu t)$ is a kernel on $[0,\infty)$ with singular point 0 and integral value 1/3, and from Lemma 2.1 that $\nu\mathrm{Ai}(\nu t)$ is a positive decreasing function of t on $[0,\infty)$ for each $\nu > 0$. Thus $\nu\mathrm{Ai}(\nu t)$ satisfies the conditions of Theorem 1.3 above on every finite interval It is difficult, however, to apply Theorem 1.3 directly here because of the infinitude of the interval, the fact that f(t) need not be integrable but merely satisfies (2.13), and the uniformity assertion. We prove Theorem 2.1, however, in almost the same way that Natanson ([24], pp. 18-20) proves Romanovski's Theorem. Natanson's Lemma 2.1 plays a major role in the proof.

Proof of Theorem 2.1

Let $\epsilon > 0$ be given. There exists an $\eta > 0$ such that $x \leq t \leq x + \eta$ implies

$$\left|h^{-1} \int_x^{x+h} [f(t) - f(x)]\ dt\right| < \epsilon,$$

and this holds for all x in $[-\delta,\delta]$, by (2.14). Now,

$$(2.16) \qquad \int_x^\infty f(t)\ \nu\ \mathrm{Ai}[\nu(t-x)]\ dt$$

$$= f\ (x) \int_x^{x+\eta} \nu\ \mathrm{Ai}[\nu(t-x)]\ dt$$

$$+ \int_x^{x+\eta} [f(t) - f(x)]\ \nu\ \mathrm{Ai}[\nu(t-x)]\ dt$$

$$+ \int_{x+\eta}^\infty f(t)\ \nu\ \mathrm{Ai}[\nu(t-x)]\ dt.$$

It is clear from (2.6) that the first term on the right of (2.16) approaches $(1/3)f(x)$, uniformly in x, $-\delta \leq x \leq \delta$, as $\nu \to \infty$. Since $\mathrm{Ai}[\nu(t-x)]$ is a positive decreasing function of t on $x \leq t \leq x+\eta$, we may apply Lemma 2.2 to get:

$$\left|\int_x^{x+\eta} [f(t) - f(x)]\ \nu\ \mathrm{Ai}\ [\nu(t-x)]\ dt\right|$$

$$< \epsilon \int_x^{x+\eta} \nu\ \mathrm{Ai}[\nu(t-x)]\ dt$$

$$< \epsilon/3, \text{ for all } \nu > 0 \text{ and all } x \in [-\delta,\delta].$$

We now use Theorem 1.1 to show that the third term on the right of (2.16) tends to zero uniformly in x. We define:

$$(2.17) \qquad \psi_\nu(t,x) = \begin{cases} 0 & , -\delta \leq t \leq x+\eta, \\ \nu \mathrm{Ai}[\nu(t-x)] \exp[\lambda(t+\delta)^{3/2}], & x+\eta < t < \infty, \end{cases}$$

for each x, $-\delta \leq x \leq \delta$, and each $\nu > 0$.

Now, by Lemma 2.1, we see that for $\nu^{3/2} \geq 3\lambda$, $Ai[\nu(t-x)] \exp[2\lambda(t-x)^{3/2}]$ is a decreasing function of t, $x + \eta \leq t < \infty$, and so

$$\nu Ai[\nu(t-x)] \exp[2\lambda(t-x)^{3/2}]$$

$$\leq \nu Ai(\nu\eta) \exp(2\lambda\eta^{3/2})$$

$$\leq [\frac{1}{\eta}\int_0^\eta \nu\ Ai(\nu t)\ dt] \exp(2\lambda\eta^{3/2}),$$

since $Ai(\nu t)$ decreases for $t > 0$. Thus we get, on using (2.6),

$$\nu Ai[\nu(t-x)] \exp[2\lambda(t-x)^{3/2}] \leq \frac{1}{3\eta} \exp(2\lambda\eta^{3/2}), \tag{2.18}$$

for $x+\eta < t < \infty$. It is clear that if the λ in condition (2.13) is ≤ 0 its replacement by a positive number only weakens that condition and thus involves no loss of generality. We suppose then that $\lambda > 0$. A simple calculation then shows that

$$\lambda(t+\delta)^{3/2} - 2\lambda(t-x)^{3/2}$$

is increasing for $x < t < (\delta+4x)/3$ and decreasing for $t > (\delta+4x)/3$. Thus we get, for $t \geq x$

$$\exp[\lambda(t+\delta)^{3/2} - 2\lambda(t-x)^{3/2}] \leq \exp[6\lambda(\frac{x+\delta}{3})^{3/2}]. \tag{2.19}$$

The Inequalities (2.18) and (2.19) taken together show that

$$|\Psi_\nu(t,x)| \leq K \quad \text{for } -\delta \leq t < \infty,\ -\delta \leq x \leq \delta, \tag{2.20}$$

and $\nu^{3/2} > 3\lambda$, where K is independent of ν, x and t.

Thus we see that condition (1.1) is satisfied in this case. To see that (1.2) holds for every finite interval $[-\delta, c]$,

we proceed as follows:

$$\int_{-\delta}^{c} \Psi_\nu(t,x)\ dt = \begin{cases} 0 , & c - x \leqslant \eta, \\ \displaystyle\int_{x+\eta}^{c} \exp[\lambda(t+\delta)^{3/2}]\,\nu\mathrm{Ai}[\nu(t-x)]\ dt, & c - x > \eta \end{cases}$$

Thus:

$$\left|\int_{-\delta}^{c} \Psi_\nu(t,x)\ dt\right| \leqslant \exp[\lambda(c+\delta)^{3/2}]\int_{\nu\eta}^{\nu(c-x)} \mathrm{Ai}(t)\ dt,$$

and this last expression tends to zero uniformly in x, as $\nu \to \infty$. Thus condition (1.2) holds in this case.

It remains to show that:

$$(2.21)\qquad V_{-\delta}^{\infty}[\Psi_\nu(t,x)] < L,$$

where L is independent of ν and x. Of course, it will be sufficient to prove this for all sufficiently large ν. (2.21) follows easily from (2.18) and (2.19) since the left-hand-sides of these inequalities when multiplied together give* $\Psi_\nu(t,x)$, the left-hand-side of (2.18) is (for large enough ν) a monotonic function of t, and the left-hand-side of (2.19) has at most one turning point. On noting that (by (2.13)):

$$\lim_{b\to\infty}\int_{-\delta}^{b} f(t)\exp[-\lambda(t+\delta)^{3/2}]\ dt \text{ exists and is finite}$$

and that $\Psi_\nu(t,x)$, as defined by (2.17) satisfies the conditions of Theorem 1.1 over the interval $(-\delta,\infty)$, we get from Theorem 1.1

$$(2.22)\qquad \lim_{\nu\to\infty}\int_{x+\eta}^{\infty} f(t)\ \nu\ \mathrm{Ai}[\nu(t-x)]\ dt = 0,$$

*for $x + \eta < t < \infty$.

uniformly in x, $-\delta \le x \le \delta$. This shows that the third term on the right of (2.16) tends to 0 uniformly in x, and so the proof of Theorem 2.1 is complete.

Theorem 2.2. Let $0 \le \delta < b < \infty$, let f be defined on $[-\delta,b]$ and let

(2.23) $$f \in BV[-\delta,b],$$

(2.24) $$f \in C[-\delta,\delta] \text{ in case } \delta > 0, \text{ and } \lim_{h\to 0+} f(\delta+h) = f(\delta).$$

Then

(2.25) $$\lim_{\nu\to\infty}\int_x^b f(t) \;\nu \,\mathrm{Ai}\,[-\nu(t-x)]\, dt = \frac{2}{3}f(x),$$

uniformly in x, $-\delta \le x \le \delta$.

Remarks. We are here concerned with a range in which the kernel is oscillatory in contrast to Theorem 2.1 where it was monotonic. Again, the case $\delta=0$ is not excluded. The theorem is of a type to which Lebesgue's Theorem 1.4 (above) might be applied but we use Lebesgue's method rather than try to show that his hypotheses are satisfied.

Proof of Theorem 2.2. Let $\varepsilon > 0$ be given. By (2.24), there exists an $\eta > 0$ such that $x \le t \le x + \eta$ implies $|f(t)-f(x)|<\varepsilon$ for each $x \in [-\delta,\delta]$.

We have:

$$\int_x^b f(t)\ \nu\,Ai\,[-\nu(t-x)]\,dt \tag{2.26}$$
$$= f(x)\int_x^{x+\eta} \nu\,Ai[-\nu(t-x)]\,dt$$
$$+\int_x^{x+\eta} [f(t)-f(x)]\ \nu\,Ai[-\nu(t-x)]\,dt$$
$$+\int_{x+\eta}^{b} f(t)\ \nu\,Ai[-\nu(t-x)]\,dt\,.$$

Now the first term on the right of (2.26) approaches $\frac{2}{3}f(x)$ uniformly in x, $-\delta \leq x \leq \delta$, as $\nu \to \infty$, by (2.7). The second mean value theorem shows that the second integral is

$$[f(x+\eta)-f(x)]\int_{\xi(\nu,x)}^{x+\eta} \nu\,Ai[-\nu(t-x)]\,dt,$$

where $x \leq \xi(\nu,x) \leq x+\eta$. (We may assume without loss of generality that $f(t)-f(x)$ is a nondecreasing bounded function, on account of (2.23).) Thus we get, on using (2.8),

$$\left|\int_x^{x+\eta} [f(t)-f(x)]\ \nu\,Ai[-\nu(t-x)]\,dt\right| \leq 2\,\varepsilon\,M,$$

for all $\nu > 0$ and all $x \in [-\delta,\delta]$. We also apply the second mean value theorem to the third integral on the right of (2.26) to get:

$$f(b)\int_{\xi(\nu,x)}^{b} \nu\,Ai[-\nu(t-x)]\,dt,\quad x+\eta \leq \xi(\nu,x) \leq b$$

i.e., $$f(b)\int_{\nu[\xi(\nu,x)-x]}^{\nu(b-x)} Ai(-t)\,dt,$$

and this clearly tends to 0 uniformly in x, as $\nu \to \infty$. (By (2.7)

$\int^{\infty} Ai(-t)\, dt$ exists). This completes the proof of Theorem 2.2.

Theorem 2.3

Let f be defined on $[b,\infty)$, $0 < b < \infty$, and let

$$t^{-3/4+\epsilon} f \in BV[b,\infty], \text{ for some } \epsilon>0. \tag{2.27}$$

Then

$$\lim_{\nu\to\infty} \int_b^{\infty} f(t)\, \nu\, Ai(-\nu t)\, dt = 0. \tag{2.28}$$

Proof of Theorem 2.3

We may assume, without loss of generality, that $\epsilon < 3/4$ since the condition (2.27) becomes weaker as ϵ decreases. Now, for $b < M < \infty$, we have:

$$\int_b^M f(t)\, \nu\, Ai(-\nu t)\, dt$$

$$= \int_b^M t^{-3/4+\epsilon} f(t)\, t^{3/4-\epsilon}\, \nu\, Ai(-\nu t)\, dt$$

$$= O(1) \int_{\xi(\nu,M)}^{M} t^{3/4-\epsilon}\, \nu\, Ai(-\nu t)\, dt, \text{ uniformly in } M,\ \nu\to\infty.$$

where $b \le \xi(\nu,M) \le M$ by (2.27), and the second mean-value theorem. Thus,

$$\int_b^M f(t)\, \nu\, Ai(-\nu t) dt = O(1)\, \nu^{-3/4+\epsilon} \int_{\nu\xi(\nu,M)}^{\nu M} \theta^{3/4-\epsilon} Ai(-\theta)\, d\theta.$$

Now $\nu^{-3/4+\epsilon} \to 0$ as $\nu\to\infty$. It follows from (2.5) that $\int^{\infty} \theta^{3/4-\epsilon} Ai(-\theta)\, d\theta$ exists and so $\int_{\nu\xi(\nu,M)}^{\nu M} \theta^{3/4-\epsilon} Ai(-\theta)\, d\theta$

approaches 0, uniformly in M, as $\nu \to \infty$. This shows that

$\int_b^{\infty} f(t)\ \nu\, Ai(-\nu t)dt$ approaches 0 as $\nu \to \infty$ and so the proof of Theorem 2.3 is complete.

We may combine theorems 2.1 and 2.2 in a single theorem:

<u>Theorem 2.4</u> Let $-\infty < a < -\delta \leq 0$, let f be defined on $[a,\infty)$ and suppose that $f \in L\ (0,b)$ for each b satisfying $0 < b < \infty$, and

(2.29) $\lim_{b\to\infty} \int_{-\delta}^{b} f(t)\ \exp[-\lambda(t+\delta)^{3/2}]\ dt$ exists and is finite for some real λ;

(2.30) $f \in BV[a,\delta]$;

(2.31) $f \in C[-\delta,\delta]$, in case $\delta \neq 0$, $\lim_{h\to 0+} f(-\delta-h) = f(-\delta)$

and $\lim_{h\to 0+} h^{-1} \int_{\delta}^{\delta+h} [f(t) - f(\delta)]\ dt = 0.$

Then

(2.32) $\lim_{\nu\to\infty} \int_a^{\infty} f(t)\ \nu\, Ai[\nu(t-x)]\ dt = f(x),$

uniformly in x, $-\delta \leq x \leq \delta$.

If we put $\delta = 0$ in Theorem 2.4 and combine it with Theorem 2.3 we get:

<u>Theorem 2.5</u> Let f be defined on $(-\infty,\infty)$ and let:

(2.33) $|t|^{-3/4+\epsilon} f \in BV[-\infty,a]$ for some $\epsilon > 0$ and some a, $-\infty < a < 0$;

(2.34) $f \in BV[a,0]$;

(2.35) $\lim_{h\to 0+} h^{-1} \int_0^h [f(t) - c]\ dt = 0.$

Suppose also that $f \in L(0,b)$, for each b satisfying $0 < b < \infty$, and that

(2.36) $\lim_{b \to \infty} \int_0^b f(t) \exp(-\lambda t^{3/2})\, dt$ exists, and is finite, for some real λ.

Then

(2.37) $$\lim_{\nu \to \infty} \int_{-\infty}^{\infty} f(t)\, \nu\, Ai(\nu t)\, dt = \frac{2}{3} f(0-) + \frac{1}{3} c.$$

CHAPTER III

KERNELS OF A MORE GENERAL TYPE

F. W. J. Olver [26] gives a technique for finding bounds for the difference between solutions of

(3.1) $$d^2w/dt^2 = [\nu^2 t + q(t)]w$$

and those of

(3.2) $$d^2w/dt^2 = \nu^2 tw,$$

where $q(t)$ is piecewise continuous on the interval considered (and satisfies a certain integrability condition in case the interval is infinite). We use a slightly simplified form of one of Olver's results to prove theorems on the convergence of singular integrals having $w(\nu,t)$ as kernel where $w(\nu,t)$ is a solution of (3.1) which behaves like the solution $\nu^{2/3}\,\mathrm{Ai}(\nu^{2/3}t)$ of (3.2). The approximation of solutions of (3.1) by Airy functions has been considered by many authors, particularly R. E. Langer. Some historical remarks on asymptotic solutions of equations like (3.1) may be found in Erdelyi [5]. Olver's form for the "error bounds", however, seems the most convenient for our applications.

Theorem 3.1

Let $q(t)$ be a continuous function of t on $a \le t < \infty$ where $-\infty < a < 0$ and let:

(3.3) $$F_1(x) = \int_x^\infty |t^{-\frac{1}{2}} q(t)|\,dt \quad \text{exist for } a \le x < \infty.$$

Then the differential equation

$$(3.4)\qquad d^2w/dt^2 = \{\nu^2 t + q(t)\}w$$

possesses a solution

$$(3.5)\qquad w(\nu,t) = \nu^{2/3}\,\mathrm{Ai}(\nu^{2/3}t) + \epsilon(\nu,t),$$

where

$$(3.6)\qquad \epsilon(\nu,t) = \nu^{2/3}\,\mathrm{Ai}\,(\nu^{2/3}\,t)\,O(\nu^{-1})$$

uniformly in t, $0 \le t < \infty$, and

$$(3.7)\qquad \epsilon(\nu,t) = O\,(\nu^{-1/3}),\ \text{uniformly in } t,\ a \le t < \infty.$$

We also have

$$(3.8)\qquad \lim_{t\to\infty} w(\nu,t)\,t^{1/4}\exp(\tfrac{2}{3}\,\nu\,t^{3/2}) = \tfrac{1}{2}\,\pi^{-1/2}\,\nu^{1/2}$$

for each $\nu > 0$, and for each $\nu > 0$, the relation (3.8) determines the solution $w(\nu,t)$ of (3.4) uniquely.

Remarks. Apart from the last assertion, Theorem 3.1 is a slightly modified form of a result of Olver ([26], Theorem 1) but since Olver's proof is given only for finite intervals [a,b] and merely outlined in the infinite case, we include the proof of Theorem 3.1 in an Appendix. Another of Olver's papers [25] is also useful in that it gives a somewhat more detailed proof of a simpler problem involving approximations in terms of exponential and trigonometric functions, rather than Airy functions.

We now prove two theorems on singular integrals involving $w(\nu,t)$.

Theorem 3.2

Let $w(\nu,t)$ be the solution of (3.4) whose existence is demonstrated in Theorem 3.1; let $f(t)$ be defined on $[-\delta,\infty)$ for some $\delta > 0$ and suppose that:

$$f(t)\exp[-\lambda(t+\delta)^{3/2}] \in L(-\delta,\infty) \tag{3.9}$$

for some real λ;

$$\lim_{h\to 0+} h^{-1}\int_x^{x+h} [f(t) - f(x)]\,dt = 0, \tag{3.10}$$

uniformly in x, $-\delta \le x \le \delta$.

Then

$$\lim_{\nu\to\infty} \int_x^{\infty} f(t)\,w(\nu,t-x)\,dt = \frac{1}{3} f(x), \tag{3.11}$$

uniformly in x, $-\delta \le x \le \delta$.

Remarks. Except for the absolute integrability in (3.9), the hypotheses here are the same as those in Theorem 2.1.

Proof of Theorem 3.2

It is clear from Theorem 2.1 that the contribution to the limit in (3.11) from the principal term in the expansion

$$w(\nu,t-x) = \nu^{2/3}\,Ai[\nu^{2/3}(t-x)] + \epsilon(\nu,t-x) \tag{3.12}$$

is $\frac{1}{3} f(x)$, uniformly in x, $-\delta \le x \le \delta$.

(3.6) shows that $\int_x^{\infty} f(t)\,\epsilon(\nu,t-x)\,dt$ exists

for all sufficiently large ν and (3.7) shows that $\int_x^b f(t)\epsilon(\nu,t-x)\,dt$ approaches 0, uniformly in x, where b is some number satisfying $\delta < b < \infty$. We also have $\int_b^\infty f(t)\epsilon(\nu,t-x)\,dt < O(\nu^{-1}) \int_b^\infty |f(t)|\,\nu^{2/3}\,\mathrm{Ai}[\nu^{2/3}(t-x)]\,dt$ and, in view of hypothesis (3.9), Theorem 2.1 shows that this last expression tends to zero, uniformly in x, as $\nu \to \infty$.

This completes the proof of Theorem 3.2.

Theorem 3.3

Let $w(\nu,t)$ be the solution of (3.4) whose existence is demonstrated in Theorem 3.1. Let f be defined on $[-\delta,b]$, $0 \leq \delta < b < \infty$, and let:

(3.13) $\quad f \in BV[-\delta,b]$,

(3.14) $\quad f \in C[-\delta,\delta]$, $f(\delta+) = f(\delta)$.

Then

(3.15) $$\lim_{\nu\to\infty} \int_x^b f(t)\, w(\nu, -t+x)\,dt = \frac{2}{3} f(x),$$

uniformly in x, $-\delta \leq x \leq \delta$.

Remark The hypotheses on f are the same as those in Theorem 2.2.

Proof of Theorem 3.3

It is clear from Theorem 2.2 that the contribution to the limit in (3.15) from the principal term in the approximation

(3.16) $$w(\nu, -t+x) = \nu^{2/3}\,\mathrm{Ai}[-\nu^{2/3}(t-x)] + \epsilon(\nu, -t+x)$$

is 2/3 f(x), uniformly in x, $-\delta \leq x \leq \delta$. The

contribution of the remainder term in (3.16) to the limit on the left of (3.15) is bounded by

$$\limsup_{\nu\to\infty} \left| \int_x^b f(t)\, \epsilon\,(\nu,t-x)\, dt \right|$$
$$< \lim_{\nu\to\infty} C\nu^{-1/3} \int_x^b |f(t)|dt$$

on account of the relation (3.7); thus the remainder term contributes 0, uniformly in x, to the limit. This completes the proof of Theorem 3.3.

<u>Theorem 3.4</u> (A combination of Theorems 3.2 and 3.3)

Let f be defined on $[a,\infty)$, where $-\infty < a < -\delta \leq 0$, and let f satisfy:

(3.17) $$f(t)\exp[-\lambda(t+\delta)^{3/2}] \in L(-\delta,\infty),$$

for some real λ;

(3.18) $$f \in BV\,[a,\delta],$$

(3.19) $$f \in C\,[-\delta,\delta],\ \lim_{h\to 0+} f(-\delta-h) = f(-\delta), \text{ and}$$

$$\lim_{h\to 0+} h^{-1} \int_\delta^{\delta+h} [f(t) - f(\delta)]\, dt = 0.$$

Let $w(\nu,t)$ be the solution of (3.4) whose existence is demonstrated in Theorem 3.1. Then

(3.20) $$\lim_{\nu\to\infty} \int_a^\infty f(t)\, w(\nu,t-x)\, dt = f(x),$$

uniformly in x, $-\delta \leq x \leq \delta$.

CHAPTER IV

SPECIAL CASES

The theorems of the previous chapter may be applied to integrals involving various special functions. We do not attempt to apply all the results of Chapter 3 here. The main idea essential to the applications of these theorems will be contained in our application of Theorem 4.1 below. Theorem 4.1 is obtained from the cases $\delta = 0$ of Theorems 3.2 and 3.3, where the condition (3.10) in Theorem 3.2 is replaced by the weaker "$f(0+)$ exists."

<u>Theorem 4.1.</u> Let $q(t)$ be a continuous function of t on $a \leq t < \infty$, where $-\infty < a < 0$ and suppose that

(4.1) $$\int_0^\infty |\theta^{-\frac{1}{2}} q(\theta)| \, d\theta \text{ exists.}$$

For each $\nu>0$, let $w(\nu,t)$ be the unique solution (cf. Theorem 3.1) of

(4.2) $$d^2w/dt^2 = \{\nu^2 t + q(t)\}w,$$

which satisfies

(4.3) $$\lim_{t\to\infty} w(\nu,t)\, t^{1/4} \exp(\tfrac{2}{3}\nu\, t^{3/2}) = \tfrac{1}{2}\pi^{-1/2}\nu^{1/2}.$$

Let f be defined on $[a,\infty)$ and suppose that:

(4.4) $$f(t) \exp(-\lambda t^{3/2}) \in L\,(0,\infty) \text{ for some real } \lambda;$$

(4.5) $$f(0+) \text{ exists;}$$

(4.6) $$f \in BV[a,0].$$

Then

(4.7) $$\lim_{\nu\to\infty} \int_a^\infty f(t)\, w(\nu,t)\, dt = \tfrac{2}{3} f(0-) + \tfrac{1}{3} f(0+).$$

Remark The continuity of q, together with (4.1) shows that

$F_1(x) = \int_x^\infty |\theta^{-\frac{1}{2}} q(\theta)|\, d\theta$ exists for $a \leq x < \infty$, so that condition (3.3) of Theorem 3.1 is satisfied.

We will use Theorem 4.1 to prove the following Theorem.

Theorem 4.2. Let $-\infty < A < x_o < B \leq \infty$, let $p(x)$ and $r(x)$ be defined on $[A,B)$ and suppose that :

(4.8) $$p(x_o) = 0,$$

(4.9) $(x-x_o)^{-1} p(x)$ is a positive twice-continuously differentiable function of x, $A \leq x < B$;

(4.10) $$\int_{x_o}^{B} \sqrt{p(t)}\, dt = \infty;$$

(4.11) $r(x)$ is a continuous function of x for $A \leq x < B$.

Let $t(x)$ be defined by:

(4.12)
$$\frac{2}{3} t^{3/2} = \int_{x_o}^{x} \sqrt{p(t)}\, dt, \quad x_o \leq x \leq B,$$

$$\frac{2}{3}(-t)^{3/2} = \int_{x}^{x_o} \sqrt{-p(t)}\, dt, \quad A \leq x \leq x_o.$$

Suppose also that

(4.13) $$p'(x) = O[\exp(\lambda_1 t^{3/2})], \quad t \to \infty, \text{ for some real } \lambda_1;$$

(4.14) $$\int_0^\infty \left| h(t) + \frac{tr(x)}{p(x)} \right| \frac{dt}{|t|^{\frac{1}{2}}} \text{ exists, where}$$

(4.15) $$h(t) = \frac{5}{16} t^{-2} + [4\, p(x)p''(x) - 5\{p'(x)\}^2] \frac{t}{16\{p(x)\}^3}.$$

For each $\nu > 0$, let $W(\nu,x)$ be the unique solution of

(4.16) $$d^2y/dx^2 = \{\nu^2 p(x) + r(x)\}y$$

which* satisfies:

(4.17) $$\lim_{x \to B-} [p(x)]^{\frac{1}{4}} W(\nu,x) \exp\left(\tfrac{2}{3}\nu t^{3/2}\right)$$
$$= \tfrac{1}{2}\pi^{-\frac{1}{2}} \nu^{\frac{1}{2}} [p'(x_o)]^{\frac{1}{2}}.$$

Suppose that $g(x)$ is defined for $A \leq x < B$, and that:

(4.18) $$g(x) \exp(-\mu t^{3/2}) \in L(x_o,B), \text{ for some real } \mu;$$

(4.19) $$g(x_o+) \text{ exists;}$$

(4.20) $$g \in BV[A,x_o].$$

Then

(4.21) $$\lim_{\nu\to\infty}\int_A^B g(x)\, W(\nu,x)\, dx = \tfrac{2}{3}g(x_o-) + \tfrac{1}{3}\, g(x_o+).$$

To facilitate the application of Theorem 4.1 to the proof of Theorem 4.2 we give the following lemma.

Lemma 4.1 Let $p(x)$ and $r(x)$ satisfy the hypotheses of Theorem 4.2. Then: (i) Under the transformations (4.12) and

(4.22) $$w = [p(x)/t]^{\frac{1}{4}} y,$$

the differential equation (4.16) becomes

(4.23) $$d^2w/dt^2 = \{\nu^2 t + h(t) + tr(x)/p(x)\}w,$$

---

*The existence and uniqueness of $W(\nu,x)$ follow from the corresponding properties of the solution $w(\nu,t)$ of (4.2) on making the transformations described in Lemma 4.1.

where $h(t)$ is given by (4.15);

(ii) $(x - x_o)^{-1}\, t(x)$ is a positive twice continuously differentiable function of x on $[A,B)$; and

(iii) $h(t)$ is continuous on $[t(A),\infty)$.

Remarks The transformation described in (i) is well known and, on making a slight change of variable, is that given in Olver [26], p. 756. (ii) and (iii) form the content of a Lemma of Olver ([26], p. 756).

Proof of Theorem 4.2 The transformation (4.12) maps the interval $A \leq x < B$ onto $t(A) \leq t < \infty$ in a one-to-one manner, the point $t = 0$ corresponding to $x = x_o$. We suppose that $t(A) = a < 0$.

We now show that the differential equation (4.23) satisfies the conditions imposed on (4.2) in Theorem 4.1. $h(t) + t\, r(x)/p(x)$ is continuous on $[a,\infty)$ because $h(t)$ is continuous there by Lemma 4.1 (iii), $r(x)$ is continuous there by (4.11), $(x-x_o)^{-1}\, p(x)$ is continuous and nonvanishing there by (4.9) and $(x-x_o)^{-1}\, t(x)$ is continuous there by Lemma 4.1 (ii). Moreover, the condition (4.14) is just (4.1) in the present case.

We see from (4.17) and Lemma 4.1 (i) that, for each $\nu > 0$

$$w(\nu,t) = [p'(x_o)]^{-1/2}[p(x)/t]^{1/4}\, W(\nu,x) \tag{4.24}$$

is the (unique) solution of (4.23) which satisfies (4.3).

We now let

(4.25) $$f(t) = g(x)\ [t/p(x)]^{3/4},\ a \leqslant t < \infty,$$

and show that f(t) satisfies (4.4), (4.5) and (4.6). First, we see by (4.18) that

$$\int_{x_0}^{B} |g(x)|\ \exp(-\mu t^{3/2})\ dx$$

exists, and since (4.12) gives

(4.26) $$dt/dx = [p(x)/t]^{1/2},$$

we find

(4.27) $$\int_0^\infty |f(t)|\ [p(x)/t]^{1/4}\ \exp(-\mu t^{3/2})\ dt \text{ exists.}$$

Now, hypothesis (4.9) shows that $(x-x_0)^{-1}\ p(x)$ is a continuous function of x on $[x_0,B)$, while Lemma 4.1 (ii) shows that $(x-x_0)^{-1}\ t(x)$ is a positive continuous function of x on $[x_0,B)$. Thus $[p(x)/t]^{1/4}$ is a continuous function of t on $[0,\infty)$. We have, for $t \to \infty$,

$$\begin{aligned}[p(x)/t]^{3/2} &\cong \frac{p'(x)}{dt/dx}\left[\frac{p(x)}{t}\right]^{1/2}, \text{ by L'Hospital's Rule,}\\ &= p'(x)\ , \text{ by (4.26),}\\ &= O[\exp(\lambda_1\ t^{3/2})], \text{ by (4.13).}\end{aligned}$$

Thus $[p(x)/t]^{1/4} = O[\exp(\frac{\lambda_1}{6}\ t^{3/2})],\ t \to \infty,$ and we see from (4.27) that

$$\int_0^\infty |f(t)|\ \exp(-\lambda t^{3/2})\ dt$$

exists when $\lambda > \mu - \lambda_1/6$. Thus (4.4) holds in the present case.

L'Hospital's Rule also shows that

(4.28) $$\lim_{t\to 0+} [p(x)/t]^{3/2} = p'(x_o)$$

so that, by (4.19), f(0+) exists, and

(4.29) $$f(0+) = g(x_o+)[p'(x_o)]^{-1/2}.$$

Hence (1.5) holds in the present case. We can show similarly that

(4.30) $$f(0-) = g(x_o-)\ [p'(x_o)]^{-1/2}.$$

Lemma 4.1 (ii) shows that $(x-x_o)^{-1}\ t(x)$ is a function of bounded variation on $[A,x_o]$ and hypothesis (4.9) shows that $(x-x_o)^{-1}\ p(x)$ is a positive function of bounded variation on $[A,x_o]$; also, by (4.20), $g \in BV[A,x_o]$. Thus f(t), considered as a function of x, is of bounded variation on $[A,x_o]$. Now x is a monotonic bounded function of t, $a \le t \le 0$, so that f(t), considered as a function of t, is of bounded variation on [a,0]. Thus (4.6) bolds in this case.

Thus all the conditions of Theorem 4.1 are satisfied when f(t) is given by (4.25), and $w(\nu,t)$ by (4.24). The conclusion of Theorem 4.1 gives:

(4.31) $$\lim_{\nu\to\infty} [p'(x_o)]^{-1/2} \int_a^\infty g(x)\,[t/p(x)]^{1/2}\ W(\nu,x)\ dt$$

$$= \frac{2}{3}\ f(0-) + \frac{1}{3} f\ (0+).$$

On using (4.26), (4.29) and (4.30), this becomes

$$\lim_{\nu\to\infty} \int_A^B g(x)W(\nu,x)dx = \frac{2}{3}\ g(x_o-) + \frac{1}{3}\ g(x_o+),$$

which is just (4.21).

This completes the proof of Theorem 4.2.

Theorem 4.3. Let $W_{\nu,m}(x)$ denote the Whittaker confluent hypergeometric function, as defined, e.g., in [35], p. 339. We suppose that m is real, and that $\nu > 0$, $x > 0$. Let $0 < A < 1$, let g(x) be defined on $[A,\infty)$ and suppose that:

(4.32) $$e^{-\mu_1 x} \; g(x) \in L(1,\infty) \text{ for some real } \mu_1;$$

(4.33) $$g(1+) \text{ exists;}$$

(4.34) $$g \in BV[A,1].$$

Then

(4.35) $$\lim_{\nu\to\infty} \frac{1}{\Gamma(\nu)} \int_A^\infty g(x)\; W_{\nu,m}(4\nu x)\, dx = \frac{2}{3}\, g(1-) + \frac{1}{3} g(1+).$$

Remarks $W_{\nu,m}(x)$ is one of the functions considered in Slater's monograph [31]. Its asymptotic properties have been discussed in great detail (see, Erdélyi and Swanson [8], Skovgaard [30], and the references given there). It may be possible to prove Theorem 4.3 by using these properties but we prefer to deduce it from Theorem 4.2.

Proof of Theorem 4.3. We let:

(4.36) $$W(\nu,x) = A_\nu\, W_{\nu,m}\,(4\,\nu\, x),\; A \leq x < \infty,\; \nu > 0$$

where

(4.37) $$A_\nu = \frac{e^\nu\, \nu^{1/2-\nu}}{(2\pi)^{1/2}}$$

We see from [8], p. 7, (2.9), that

$$y = W(\nu, x)$$

satisfies

$$(4.38)\qquad \frac{d^2y}{dx^2} = \left\{4\nu^2\left(1 - \frac{1}{x}\right) + \frac{m^2 - 1/4}{x^2}\right\}y,$$

and we proceed to show that the hypotheses of Theorem 4.2 are satisfied, with

$$x_o = 1,\ B = \infty,\ p(x) = 4\left(1 - \frac{1}{x}\right),\ r(x) = \frac{m^2 - 1/4}{x^2}.$$

Conditions (4.8) to (4.11) are immediate. The transformation (4.12) for this special case has already been considered (except for a multiplicative constant) in [8] and [30]. The following are easily proven (and may be found in [8], p. 26, (8.8) and (8.9)):

$$(4.39)\qquad \frac{2}{3}t^{3/2} = 2(x^2 - x)^{\frac{1}{2}} - 2\log\left[x^{\frac{1}{2}} + (x-1)^{\frac{1}{2}}\right],\ 1 \leq x < \infty;$$

$$(4.40)\qquad \frac{2}{3}t^{3/2} = 2x - \log 4x - 1 + O(x^{-\frac{1}{2}}),\ x \to \infty.$$

We have $p'(x) = 4x^{-2}$, so that it is clear from (4.40) that (4.13) holds. We also have, in the present case:

$$(4.41)\qquad h(t) + tr(x)/p(x)$$

$$= \frac{5}{16}t^{-2} - \frac{(8x-3)t}{64x(x-1)^3} + \frac{(4m^2-1)t}{16x(x-1)},\ A < x < \infty,$$

$$= \frac{5}{16}t^{-2} + O(1)t^{-3/2} + O(1)(t^{-2}),\ t \to \infty,$$

by using (4.40). Using this asymptotic result and the continuity of the left-hand side of (4.41)

we see that

$$\int_0^\infty \left| h(t) + \frac{tr(x)}{p(x)} \right| \frac{dt}{|t|^{\frac{1}{2}}}$$

exists and is finite, i.e.,(4.14) holds in this case.

We now use ([35], p. 343):

(4.42) $$W_{k,m}(x) \cong e^{-\frac{1}{2}x} x^k, \quad x \to \infty,$$

together with (4.40), to see that

(4.43) $$\lim_{x\to\infty} [4(1 - \frac{1}{x})]^{\frac{1}{4}} W(\nu,x) \exp(\tfrac{2}{3}\nu\, t^{3/2}) = \pi^{-\frac{1}{2}} \nu^{\frac{1}{2}},$$

which is just the condition (4.17) for this case.

We are given, (4.32):

$$e^{-\mu_1 x}\, g(x) \quad \varepsilon\ L(1,\infty)$$

so it is clear that if we choose $3\mu > \max(\mu_1, 0)$,

we get from (4.40):

$$e^{-\mu t^{3/2}} = O(e^{-\mu_1 x}), \quad x \to \infty$$

and so

$$e^{-\mu t^{3/2}} \quad g(x) \quad \varepsilon\ L(1,\infty).$$

Thus condition (4.18) holds in the present case. (4.19) and (4.20) are just (4.33) and (4.34) respectively. Hence all the hypotheses of Theorem 4.2 are satisfied and we get

$$\lim_{\nu\to\infty} A_\nu \int_A^\infty g(x)\, W_{\nu,m}(4\,\nu\, x)\, dx$$
$$= \frac{1}{3}\, g(1-) + \frac{2}{3}\, g(1+).$$

Now, Stirling's formula shows that

$$\lim_{\nu\to\infty} A_\nu \Gamma(\nu) = 1,$$

so that we immediately get (4.35).

This completes the proof of Theorem 4.3.

Corollary (A kernel involving the Laguerre polynomials)
The Laguerre polynomials $L_n^{(\alpha)}(x)$ are related to $W_{\nu,m}(x)$ in the following way (cf. Slater [30], p. 95):

$$(4.44)\qquad L_n^{(\alpha)}(x) = \frac{(-1)^n}{n!} e^{\frac{1}{2}x} x^{-\frac{1}{2}(\alpha+1)} W_{\frac{\alpha+1}{2}+n,\, \frac{\alpha}{2}}(x)$$

Using (4.44) we can show that, if g(x) satisfies the hypotheses of Theorem 4.3, then, for each real x

$$(4.45)\qquad \lim_{n\to\infty} (-1)^n\, n\, 2^{\alpha+1} \int_A^\infty g(x)\, e^{-2\nu x} L_n^{(\alpha)}(4\nu x)\, dx$$

$$= \frac{2}{3} g(1-) + \frac{1}{3} g(1+),$$

where $\nu = n + \dfrac{\alpha+1}{2}$.

Theorem 4.4 Let $D_\nu(x)$ be the parabolic cylinder function, as defined, e.g., in [6], vol. 2, Chapter 8. Let ν be real, $-1 < A < 1$, let g(x) be defined on $[A,\infty)$, and suppose that:

$$(4.46)\qquad e^{-\mu x^2} g(x) \in L(1,\infty) \text{ for some real } \mu;$$

$$(4.47)\qquad g(1+) \text{ exists;}$$

$$(4.48)\qquad g \in BV[A,1].$$

Then

$$(4.49)\qquad \lim_{\nu\to\infty} \frac{2^{(3-\nu)/2}\, \nu^{1/4}}{\Gamma(\nu/2)} \int_A^\infty g(x)\, D_{\nu-\frac{1}{2}}(2x\sqrt{\nu})\, dx$$

$$= \frac{2}{3} g(1-) + \frac{1}{3} g(1+).$$

Remarks. Erdélyi, Kennedy and McGregor [7] use the differential

equation satisfied by $D_\nu(x)$ to derive asymptotic forms for this function. A differential equations approach has been used by H. Skovgaard [29] to derive asymptotic forms of Hermite polynomials (which are special parabolic cylinder functions). The proof of Theorem 4.4 is very similar to that of Theorem 4.3 so we merely give it in outline.

Proof of Theorem 4.4 (Outline) We let

$$W(\nu,x) = B_\nu D_{\nu-\frac{1}{2}}(2x\sqrt{\nu}), \tag{4.50}$$

where

$$B_\nu = \sqrt{\frac{2}{\pi}}\, e^{\nu/2}\, \nu^{-\nu/2 + 3/4}. \tag{4.51}$$

We see from [7], p. 462 that $y = W(\nu,x)$ satisfies:

$$d^2 y/dx^2 = 4\nu^2(x^2-1)y, \tag{4.52}$$

and we proceed to show that the hypotheses of Theorem 4.2 are satisfied, where:

$$x_0 = 1,\ B = \infty,\ p(x) = 4(x^2-1),\ r(x) = 0.$$

Conditions (4.8) to (4.11) are immediate, as before; the transformation (4.12) in this special case is (except for a multiplicative constant) the one considered in [29], p.4. Either directly, or from [29], p.5, (2.17) we get

$$\frac{2}{3}t^{3/2} = x^2 - \log 2x - \frac{1}{2} + O(x^{-2}),\ x \to \infty. \tag{4.53}$$

This asymptotic formula, together with Lemma 4.1 (iii),

enables us to prove (4.13) and (4.14) in the present case.

We now use ([35], p. 348):

(4.54) $$D_\nu(x) \cong x^\nu e^{-x^2/4}, \quad x \to \infty,$$

together with (4.53), to get

(4.55) $$\lim_{\nu\to\infty} [4(x^2-1)]^{1/4} W(\nu,x) \exp(\tfrac{2}{3}\nu t^{3/2}) = \sqrt{2}\pi^{-\frac{1}{2}}\nu^{\frac{1}{2}}$$

which is just (4.17) for this case.

With the aid of (4.53) we show that (4.46) implies (4.18), and (4.19), (4.20) are just (4.47), (4.48) respectively.

Thus we see that all the hypotheses of Theorem 4.2 are satisfied and we get (with B_ν given by (4.51)):

(4.56) $$\lim_{\nu\to\infty} B_\nu \int_A^\infty g(x)\, D_{\nu-\frac{1}{2}}(2x\sqrt{\nu})\, dx$$
$$= \frac{2}{3} g(1-) + \frac{1}{3} g(1+).$$

On noting that, by Stirling's formula,

$$\lim_{\nu\to\infty} \frac{B_\nu\, \Gamma(\nu/2)}{2^{3-\nu/2}\, \nu^{1/4}} = 1,$$

we get (4.49). This completes the proof of Theorem 4.4.

<u>Corollary</u> (A kernel involving Hermite polynomials).

The parabolic cylinder function $D_n(z)$ is related to the Hermite polynomial $H_n(z)$ in the following way (cf. [6], vol. 2, p. 117(9)):

(4.57) $$D_n(z) = e^{-z^2/4}\, 2^{-n/2}\, H_n(2^{-1/2} z)$$

Using (4.57) we can show that, under the hypotheses of Theorem 4.4:

(4.58) $$\lim_{n\to\infty} \frac{2^{-7/4-\nu}\,\nu^{\frac14}}{\Gamma(\nu/2)} \int_A^\infty g(x)\, e^{-\nu x^2}\, H_n(x\sqrt{2\nu})\, dx$$

$$= \frac{2}{3}\, g(1-) + \frac{1}{3}(g(1+)\,, \quad \nu = n + 1/2.$$

Theorem 4.5. (A kernel involving a Bessel function)

Suppose that $1 < b < \infty$, that $g(x)$ is defined on $(0,b]$, and that:

(4.59) $x^\mu g(x) \in L(0,1)$, for some real μ;

(4.60) $g(1-)$ exists;

(4.61) $g \in BV[1,b]$.

Then

(4.62) $$\lim_{\nu\to\infty} \int_0^b g(x)\, \nu J_\nu(\nu x)\, dx = \frac{1}{3} g(1-) + \frac{2}{3}\, g(1+).$$

Remarks. This is the result of Lorch and Szego ([20](a), Theorem 1) referred to in our Introduction. They prove it by using certain properties of Bessel functions; we will prove it by using Theorem 4.2. The proof is very similar to that of Theorems 4.3 and 4.4, one difference being that the oscillatory region of the kernel is now to the right of the point $t = 1$.

Proof of Theorem 4.5 (outline). We let

(4.63) $$W(\nu,x) = C_\nu (-x)^{\frac12} J_\nu(-\nu x),$$

where

(4.64) $$C_\nu = \frac{\Gamma(\nu+1)\, e^\nu}{(2\pi)^{\frac12}\, \nu^{\nu-\frac12}}.$$

If $y = W(\nu,x)$, then y satisfies the differential equation:

$$\frac{d^2y}{dx^2} = [\nu^2(x^{-2}-1) - x^{-2}/4]y, \tag{4.65}$$

(cf. [5], p. 20, (5.2)). We proceed to show that the hypotheses of Theorem 4.2 are satisfied, where:

$$p(x) = x^{-2} - 1,\ r(x) = -\tfrac{1}{4}x^{-2},\ x_o = -1,\ B = 0.$$

The hypotheses (4.8) to (4.11) are easily shown to hold in this case.

The transformation (4.12) is in this case:

$$(4.66)\qquad \begin{aligned} \tfrac{2}{3}t^{3/2} &= \int_{-1}^{x} \sqrt{\frac{1-t^2}{t^2}}\,dt,\ -1 \le x < 0, \\ \tfrac{2}{3}(-t)^{3/2} &= \int_{x}^{-1} \sqrt{\frac{t^2-1}{t^2}}\,dt,\ A \leq x \leq -1 \end{aligned}$$

where we have chosen $A = -b$. Apart from some changes in sign this is the transformation discussed in [5], p. 20, and we may deduce from the work there, or from (4.66) directly, that:

$$\begin{aligned} \tfrac{2}{3}t^{3/2} &= -(1-x^2)^{1/2} + \log\,[1 + (1-x^2)^{1/2}] \\ &\quad - \log(-x),\ -1 \leq x < 0, \end{aligned} \tag{4.67}$$

$$\tfrac{2}{3}t^{3/2} = -\log(-x) + \log 2 - 1 + O(x^2),\ x \to 0-. \tag{4.68}$$

We have $p'(x) = -2x^{-3}$; this gives, on using (4.68)

$$p'(x) = O(1)\exp\,(2t^{3/2}),\ x \to 0-,$$

and so (4.13) is satisfied in the present case.

We also have

$$h(t) + tr(x)/p(x) = \frac{5}{16}\, t^{-2} + \frac{tx^2(x^2+4)}{4(x^2-1)^3}, \tag{4.69}$$

(cf. [27], p.207). On using (4.68), this gives:

$$h(t) + tr(x)/p(x) = \frac{5}{16}t^{-2} + O(1)t \exp[-\tfrac{4}{3}t^{3/2}], t\to\infty; \tag{4.70}$$

hence, the integrability condition (4.14) is satisfied. We now use

$$J_\nu(x) \cong \frac{(x/2)^\nu}{\Gamma(\nu+1)}, \quad x \to 0+, \tag{4.71}$$

together with (4.68), to get:

$$\lim_{x\to 0-} [x^{-2}-1]^{\frac{1}{4}} W(\nu,x) \exp(\tfrac{2}{3}\nu t^{3/2}) = (2\pi)^{-\frac{1}{2}}\nu^{\frac{1}{2}}; \tag{4.72}$$

this is just (4.17) for the present case, since $p'(-1) = 2$.

We now define a function $g_1(x)$ by

$$(-x)^{\frac{1}{2}} g_1(x) = g(-x), \quad A \leq x < 0. \tag{4.73}$$

It is clear from (4.59) that

$$(-x)^{\mu+\frac{1}{2}} g_1(x) \in L(-1,0); \tag{4.74}$$

it follows, then, from (4.68), that if we choose $\frac{3}{2}\lambda > \max(\mu+\frac{1}{2},0)$ we will obtain:

$$g\;(x) \exp(-\lambda t^{3/2}) \in L(-1,0). \tag{4.75}$$

Thus (4.18) holds in this case. It is clear from (4.60) and (4.61) that

$$g_1(-1-) \text{ exists}, \tag{4.76}$$

and

$$g_1 \in BV[A,-1] \tag{4.77}$$

Thus all the hypotheses of Theorem 4.2 are satisfied when

$W(\nu,x)$ is given by (4.63) and $g_1(x)$ is given by (4.73).

The conclusion of Theorem 4.2 gives:

$$(4.78) \qquad \lim_{\nu\to\infty} C_\nu \int_A^0 g_1(x)(-x)^{\frac{1}{2}} J_\nu(-\nu x)\, dx = \frac{2}{3}\, g_1(-1-) + \frac{1}{3} g_1(-1+),$$

i.e.,

$$(4.79) \qquad \lim_{\nu\to\infty} C_\nu \int_0^b g(x)\, J_\nu(\nu x)\, dx = \frac{1}{3} g(1-) + \frac{2}{3} g(1+).$$

Stirling's formula shows that

$$\lim_{\nu\to\infty} C_\nu/\nu = 1,$$

so that (4.62) follows immediately from (4.79). This completes the proof of Theorem 4.5.

We have now proved or stated theorems on singular integrals involving the following special functions:

The Airy function $Ai(x)$ (Theorems 2.1 to 2.5);

The Whittaker function $W_{k,m}(x)$ (Theorem 4.3);

The Laguerre polynomial $L_n^{(\alpha)}(x)$ (Corollary to Theorem 4.3);

The parabolic cylinder function $D_\nu(x)$ (Theorem 4.4);

The Hermite polynomial $H_n(x)$ (Corollary to Theorem 4.4);

The Bessel function $J_\nu(x)$ (Theorem 4.5).

CHAPTER V

CESÀRO SUMMABILITY

In order to prove a convergence theorem for

$$\int_a^\infty f(t)\, \nu^{2/3}\, Ai[\nu^{2/3}(t-x)]dt$$

we have so far assumed f to be of bounded variation over the interval in which Ai(t-x) is oscillatory. We shall see in this chapter that this condition may be considerably weakened if, instead of convergence, we consider Cesàro (C,k) summability, $k \geq 1/2$. The definition of Cesàro summability which we will use is the following:

Definition 5.1 (cf. Hardy [14], pp. 110-111).

Let $F(\nu)$ be a bounded integrable function of ν on each interval $(0,A)$, $0 < A < \infty$. We define $C_{k,\nu}\{F(\nu)\}$, the (C,k) mean of $F(\nu)$, in the following way:

$$(5.1)\qquad \begin{aligned} C_{0,\nu}\{F(\nu)\} &= F(\nu), \\ C_{k,\nu}\{F(\nu)\} &= k\nu^{-k}\int_0^\nu (\nu-p)^{k-1}\, F(p)\, dp,\ k > 0. \end{aligned}$$

We say that

$$(5.2)\qquad (C,k) - \lim_{\nu\to\infty} F(\nu) = A,$$

if

$$\lim_{\nu\to\infty} C_{k,\nu}\{F(\nu)\} = A.$$

Remarks. Hardy (loc. cit.) gives the definition for the case in which $F(\nu)$ is a definite integral with upper limit of integration ν. He remarks ([14], p. 119) that it may be

extended to integrable functions $F(\nu)$. The definition is analogous to that of Cesàro summability for sequences and series, and apparently for this reason, separate proofs of the properties of (C,k) summability (in the sense of Definition (5.1)) are rare. Hence, we include a proof of the "regularity" of this type of summability (Lemma 5.1, below).

We will show in this chapter that, for $k>1/2$, and with $0 < b < \infty$,

$$(5.3) \qquad (C,k)\text{-}\lim_{\nu\to\infty}\int_0^b f(t)\,\nu^{2/3}\,\mathrm{Ai}(-\nu^{2/3}t)\,dt = \frac{2}{3}\,f(0+)$$

under the assumptions:

$$(5.4) \qquad f \in L[0,b],\ f(0+) \text{ exists.}$$

These are milder assumptions than the assumption $f \in BV[0,b]$, which is necessary to deduce the case $k=0$ of (5.3) from Theorem 2.2. We show also that conditions (5.4) are not sufficient to imply (5.3) in case $0 \leq k \leq 1/2$. The above results, which form the content of Theorem 5.1, are completely analogous to results of Lorch and Szego concerning the kernel $\nu J_\nu(\nu t)$ ([21], Theorem 4.1). Our method of proof is almost exactly the same as that of these authors.

We show in Theorem 5.2 (below) that (5.3) holds even in case $k = 1/2$, provided f satisfies, in addition to (5.4), a growth condition on the right of the point $t = 0$, but that conditions (5.4), together with this extra condition, are not sufficient to imply (5.3) in case $0 \leq k < 1/2$.

In Theorem 5.3, we extend some of these results to singular integrals involving $w(\nu,t)$ (in the notation of Chapter 3).

We find it necessary however to make an extra assumption concerning the existence of the integrals involved in the (C,k) process.

The proofs of Theorems 5.1 and 5.2 will be made to depend on several lemmas.

Lemma 5.1 (Regularity of the (C,k) - method).

Let $F(\nu)$ be a bounded, integrable function of ν on each interval $[0,b]$, $0 < b < \infty$. If, for a fixed $k > 0$,

(5.5) $$(C,k) - \lim_{\nu\to\infty} F(\nu) = A,$$

then

(5.6) $$(C,k+\alpha) - \lim_{\nu\to\infty} F(\nu) = A, \text{ for each } \alpha > 0.$$

Proof We use the method suggested in Hardy [14], p.111. It will clearly be sufficient to consider the case $A = 0$. We now use:

(5.7) $$\int_0^\nu (\nu-p)^{\alpha-1} \left[\int_0^p (p-q)^{k-1} F(q)\, dq\right] dp$$
$$= B(\alpha,k) \int_0^\nu (\nu-p)^{k+\alpha-1} F(p)\, dp, \quad k > 0, \quad \alpha > 0.$$

This relation is given in Hardy [14], p. 111, (5.14.3), and easily proved by interchanging the order of the integrations on the left-hand-side. On using (5.1) and (5.7) we find:

(5.8) $$C_{k+\alpha,\nu}\{F(\nu)\} = \frac{(k+\alpha)\,\nu^{-k-\alpha}}{kB(\alpha,k)} \int_0^\nu (\nu-p)^{\alpha-1} p^k C_{k,p}\{F(p)\}dp.$$

Now, if $\epsilon > 0$ be given, we can, by (5.5), with $A = 0$, choose N so large that $p \geq N$ implies

$$|C_{k,p}\{F(p)\}| < \epsilon.$$

We then have

$$|\nu^{-k-\alpha}\int_N^\nu (\nu-p)^{\alpha-1}\ p^k\ C_{k,p}\{F(p)\}dp|$$

$$\leq \epsilon\ \nu^{-k-\alpha}\int_0^\nu (\nu-p)^{\alpha-1}\ p^k\ dp$$

$$= \epsilon\ B(\alpha,k+1).$$

We also have

$$\int_0^N (\nu-p)^{\alpha-1}\ p^k\ C_{k,p}\{F(p)\}dp$$

$$= k\int_0^N (\nu-p)^{\alpha-1}[\int_0^p (p-q)^{k-1}\ F(q)\ dq]\ dp$$

$$= k\ B(k,\alpha)\int_0^N (\nu-q)^{\alpha+k-1}\ F(q)\ dq,$$

the last equality following from [6], vol. 1, p.10, (13).

This gives:

$$\nu^{-k-\alpha}\int_0^N (\nu-p)^{\alpha-1}\ p^k\ C_{k,p}\{F(p)\}\ dp = O(\nu^{-1})\int_0^N |F(q)|dq$$

and so we have

$$\lim_{\nu\to\infty} \nu^{-k-\alpha}\int_0^\nu (\nu-p)^{\alpha-1}\ p^k\ C_{k,p}\{F(p)\}\ dp = 0.$$

It is then clear from (5.8) that

$$(C,k+\alpha) - \lim_{\nu\to\infty} F(\nu) = 0,$$

so that (5.6) holds (with A=0) for k > 0. The corresponding result for k = 0 is proved similarly except that (5.8) is replaced by

$$C_{\alpha,\nu}\{F(\nu)\} = \alpha\nu^{-\alpha}\int_0^\nu (\nu-p)^{\alpha-1}\ F(p)\ dp.$$

This completes the proof of Lemma 5.1.

Lemma 5.2. Let $F(\nu,t)$ be a bounded, integrable function of ν on each interval $[0,b]$, $0 < b < \infty$, and for each t in a set E of

real numbers. Suppose that:

(5.9) $$\lim_{\nu\to\infty} F(\nu,t) = 0,$$

uniformly in t, $t \in E$;

(5.10) $$\left|\int_\alpha^\beta F(\nu,t)d\nu\right| \leq M(b),$$

for all $t \in E$, and for all subintervals $[\alpha,\beta]$ of $[0,b]$.

Then, for $k > 0$,

(5.11) $$(C,k) - \lim_{\nu\to\infty} F(\nu,t) = 0,$$

uniformly in t, $t \in E$.

Proof. Let k be a fixed positive number; we have

(5.12) $$C_{k,\nu}\{F(\nu,t)\} = k\nu^{-k}\int_0^\nu (\nu-p)^{k-1} F(p,t)\, dp.$$

If $\epsilon > 0$ is given, then (on account of (5.9)) we can find N such that

$$|F(p,t)| < \epsilon,$$

whenever $p \geq N$, and $t \in E$. Thus, we have

(5.13) $$\left|k\nu^{-k}\int_N^\nu (\nu-p)^{k-1} F(p,t)\, dp\right| < \epsilon,$$

for all $t \in E$. We also have, for $0 < k \leq 1$,

$$k\nu^{-k}\int_0^N (\nu-p)^{k-1} F(p,t)\, dp$$

$$=k\nu^{-k} (\nu-N)^{k-1} \int_{\xi(\nu,t)}^N F(p,t)\, dp,$$

by the second mean-value theorem, where $0 \leq \xi(\nu,t) \leq N$. We see from (5.10), that this last expression is bounded by

$$k\nu^{-1}[1 - N/\nu]^{k-1} M(N),$$

and so approaches zero, uniformly in t, as $\nu \to \infty$. If $k > 1$,

we have

$$k\nu^{-k}\int_0^N (\nu-p)^{k-1}\; F(p,t)\; dp = k\nu^{-1}\int_0^{\eta(\nu,t)} F(p,t)dp,$$

where $0 \leq \eta(\nu,t) \leq N$, and the right-hand-side here again tends to zero, uniformly in t, as $\nu \to \infty$, by (5.10). Recalling (5.13), we now see that

$$k\nu^{-k}\int_0^{\nu} (\nu-p)^{k-1}\; F(p,t)\; dp$$

tends to 0, uniformly in t, as $\nu \to \infty$, and so the proof of Lemma 5.2 is complete.

<u>Lemma 5.3</u>. Let k and b be fixed, $0 \leq k < 1$, $0 < b < \infty$. Then

(5.14) $$C_{k,\nu}\{\nu^{2/3}\; Ai(-\nu^{2/3}t)\}$$

$$= a_k\; t^{-1/4-3k/2}\nu^{-k+1/2}\; \cos(\tfrac{2}{3}\nu t^{3/2}-\tfrac{1}{2}k\pi - \tfrac{\pi}{4})$$

$$+ t^{-7/4}\; O(\nu^{-1/2}) + t^{-1/4}\; O(\nu^{-1/2}),\quad 0 < t \leq b,$$

the O-terms holding uniformly in t, $0 < t \leq b$, where

(5.15) $$a_k = \pi^{-1/2}\; \Gamma(k+1)(3/2)^k.$$

<u>Proof</u>. Throughout the proof all O-terms hold uniformly in t, $0 < t \leq b$, as $\nu \to \infty$.

We see from (2.5) that

(5.16) $$\nu^{2/3}Ai(-\nu^{2/3}t) = \pi^{-1/2}\nu^{\frac{1}{2}}t^{-\frac{1}{4}}\; \cos(\tfrac{2}{3}\nu t^{3/2}-\tfrac{\pi}{4}) + \mathcal{R}(\nu,t),$$

where

(5.17) $$|\mathcal{R}(\nu,t)| \leq C\nu^{-1/2}\; t^{-7/4},\quad \nu > 0,\; t > 0,$$

for some constant $C > 0$. Thus we see that (5.14) holds in the case $k = 0$. When $k > 0$ and $t > 0$, $C_{k,\nu}\{\mathcal{R}(\nu,t)\}$ exists since

the (C,k) means of the other two expressions in (5.16) exist. On account of (5.17), we have

(5.18) $$C_{k,\nu}\{\mathcal{R}(\nu,t)\} \leq C\, k\, B(k,1/2)\,\nu^{-1/2}\, t^{-7/4}, \quad \nu > 0,\ t > 0,$$

and so we get, from (5.16):

(5.19) $$C_{k,\nu}\{\nu^{2/3}\,\mathrm{Ai}(-\nu^{2/3}t)\}$$
$$= k\pi^{-1/2}\, t^{-1/4}\nu^{-k} \int_0^{\nu} (\nu-p)^{k-1}$$
$$p^{1/2}\cos(\tfrac{2}{3}pt^{3/2} - \tfrac{\pi}{4})\, dp + t^{-7/4}\, O(\nu^{-1/2}).$$

The error made in replacing $p^{1/2}$ by $\nu^{1/2}$ in the integral in (5.19) is bounded by

(5.20) $$\left|k\pi^{-1/2}\, t^{-1/4}\nu^{-k} \int_0^{\nu} (\nu^{1/2}-p^{1/2})(\nu-p)^{k-1} \cos(\tfrac{2}{3}\, pt^{3/2} - \tfrac{\pi}{4})\, dp\right|.$$

Now,

(5.21) $$(\nu^{1/2} - p^{1/2})(\nu-p)^{k-1} = (\nu^{1/2}-p^{1/2})^{k}(\nu^{1/2}+p^{1/2})^{k-1}.$$

Since $0 \leq k < 1$, we see that both factors on the right-hand-side of (5.21) are nonincreasing functions of p, $0 \leq p \leq \nu$. Thus, we can apply the second mean-value theorem to the integral in (5.20) to see that the error referred to above is bounded by $Kt^{-1/4}\nu^{-1/2}$, where K is some constant which depends only on k. Thus, (5.19) becomes

(5.22) $$C_{k,\nu}\{\nu^{2/3}\,\mathrm{Ai}(-\nu^{2/3}\, t)\}$$
$$= k\pi^{-1/2}t^{-1/4}\nu^{-k+1/2}\int_0^{\nu} (\nu-p)^{k-1} \cos(\tfrac{2}{3}\, pt^{3/2} - \pi/4)\,dp$$
$$+ t^{-7/4}\, O(\nu^{-1/2}) + t^{-1/4}\, O(\nu^{-1/2}).$$

Now,

$$(5.23)\qquad \int_0^{\nu}(\nu-p)^{k-1}\cos(\tfrac{2}{3}pt^{3/2}-\pi/4)\,dp$$

$$= (3/2)^k\, t^{-3k/2}\int_0^{(2/3)\nu t^{3/2}} x^{k-1}\cos(x-\tfrac{2}{3}\nu t^{3/2}+\pi/4)\,dx$$

$$=(3/2)^k t^{-3k/2}\int_0^{\infty} x^{k-1}\cos(x-\tfrac{2}{3}\nu t^{3/2}+\pi/4)\,dx + t^{-3/2}\nu^{k-1}\,O(1)$$

$$=(3/2)^k t^{-3k/2}\Gamma(k)\cos(\tfrac{2}{3}\nu t^{3/2}-k\pi/2-\pi/4) + t^{-3/2}\nu^{k-1}O(1);$$

the second equality follows from an application of the second mean-value theorem, and the third equality follows on using equations (37) and (38) in [6], vol. 1, p. 13. If we substitute this result in (5.22), we obtain (5.14), and Lemma 5. 3 is proven.

<u>Lemma 5.4</u>. We again let $0 < b < \infty$, and we suppose that:

$$(5.24)\qquad \Lambda_k(\nu,\alpha) = \int_0^b t^{\alpha}|C_{k,\nu}\{\nu^{2/3}Ai(-\nu^{2/3}t)\}|\,dt,$$

for each $\nu > 0$, $\alpha \geqslant 0$ and $0 \leqslant k < 1$. Then, as $\nu \to \infty$,

$$(5.25)\qquad \Lambda_k(\nu,\alpha) = c_k(\alpha)\nu^{1/2-k}+O(1),\quad 0 \leqslant k < 1/2,$$

where

$$(5.26)\qquad c_k(\alpha) = 2\pi^{-3/2}(3/2)^{k-1}\Gamma(k+1)b^{\alpha+3/4-3k/2}/(\tfrac{2}{3}\alpha+\tfrac{1}{2}-k);$$

$$(5.27)\qquad \Lambda_{1/2}(\nu,0) = \pi^{-1}(2/3)^{1/2}\log\nu + O(1);$$

$$(5.28)\qquad \Lambda_{1/2}(\nu,\alpha) = O(1),\quad \alpha > 0;$$

and

$$(5.29)\qquad \Lambda_k(\nu,\alpha) = O(1),\quad 1/2 < k < 1.$$

<u>Proof</u> In this proof all O-terms will refer to $\nu \to \infty$.
We have, for all α and k considered,

$$(5.30) \qquad \Lambda_k(\nu,\alpha) = \int_{\nu^{-2/3}}^{b} t^{\alpha} |C_{k,\nu}\{\nu^{2/3}Ai(-\nu^{2/3}t)\}| dt + O(1),$$

because Ai(t) is bounded (from (2.2), for example), and hence

$$|C_{k,\nu}\{\nu^{2/3}\ Ai(-\nu^{2/3}t)\}|$$
$$\leqslant K\ k\nu^{-k}\int_0^{\nu} (\nu-p)^{k-1}\ p^{2/3}\ dp, \text{ for some constant } K,$$
$$\leqslant K_1\ \nu^{2/3}, \text{ for some constant } K_1.$$

We now use the result (5.14) of Lemma 5.3. The contributions from the remainder terms in (5.14) to the integral in (5.30) are O(1). Thus (5.30) becomes:

$$(5.31) \qquad \Lambda_k(\nu,\alpha) = a_k\nu^{1/2-k} \int_{\nu^{-2/3}}^{b} t^{\alpha-1/4-3k/2} \left|\cos(\tfrac{2}{3}\nu t^{3/2} - \tfrac{1}{2}k\pi - \pi/4)\right| dt + O(1),$$

a_k being given by (5.15).
Making the change of variable

$$(5.32) \qquad \tfrac{2}{3}\ t^{3/2} = \theta$$

and writing

$$(5.33) \qquad \beta = 2\alpha/3 - \tfrac{1}{2} - k,$$

we find

$$(5.34) \quad \Lambda_k(\nu,\alpha) = (3/2)^{\beta}\ a_k\ \nu^{-k+1/2}\int_{\frac{2}{3}\nu^{-1}}^{\frac{2}{3}b^{3/2}} \theta^{\beta}\left|\cos(\nu\theta - (1/2)k\pi - \pi/4)\right|d\theta + O(1).$$

The error made in replacing $\left|\cos(\nu\theta - (1/2)k\pi - \pi/4)\right|$ by its mean value $2/\pi$ in the integral in (5.34) is

$$O(1)\cdot \nu^{-2\alpha/3}\int_{2/3}^{(2/3)\nu b^{3/2}} x^{\beta}\{|\cos(x-(1/2)k\pi-\pi/4)| - \tfrac{2}{\pi}\}dx,$$

and this is O(1), by the second mean-value theorem. From (5.34), we obtain:

$$(5.35)\qquad \Lambda_k(\nu,\alpha) = (\tfrac{2}{\pi})(\tfrac{3}{2})^{\beta}\; a_k\nu^{1/2-k}\int_{\frac{2}{3}\nu^{-1}}^{(2/3)b^{3/2}} \theta^{\beta}d\theta + O(1),$$

and so

$$(5.36)\qquad \Lambda_k(\nu,\alpha) = \frac{4a_k}{3\pi(\beta+1)}\,[b^{3(\beta+1)/2} - \nu^{-\beta-1}]\nu^{1/2-k} + O(1),\ \beta \neq -1,$$

$$(5.37)\qquad \Lambda_k(\nu,\alpha) = \frac{4a_k}{3\pi}\,[\tfrac{3}{2}\log b + \log \nu]\,\nu^{1/2-k} + O(1), \beta = -1.$$

When $0 \leq k < 1/2$ and $\alpha \geq 0$, we have $\beta > -1$, so we can use (5.36) to get (5.25), with $c_k(\alpha)$ given by (5.26).

When $k = 1/2$ and $\alpha = 0$, we have $\beta = -1$, so we use (5.37) to get (5.27).

When $k = 1/2$ and $\alpha > 0$ we have $\beta > -1$ so we use (5.36) to get (5.28).

When $1/2 < k < 1$, either the case $\beta = -1$ or $\beta \neq -1$ can arise depending on whether k is, or is not equal to $\frac{1}{2} + 2\alpha/3$. In the first case, (5.29) follows from (5.37), while in the second it follows from (5.36).

This completes the proof of Lemma 5.4.

<u>Theorem 5.1</u> (i) Let f be defined on $[0,b]$, $0 < b < \infty$ and suppose that:

$$(5.38)\qquad f \in L(0,b),$$

$$(5.39)\qquad f(0+) \text{ exists.}$$

Then

$$(5.40)\qquad (C,k) - \lim_{\nu\to\infty}\int_0^b f(t)\,\nu^{2/3}\,Ai(-\nu^{2/3}t)\,dt = \tfrac{2}{3}f(0+),$$

for $k > 1/2$.

(ii) There exists a continuous function f on $[0,b]$ for which

$$C_{1/2,\nu}\{\int_0^b f(t)\ \nu^{2/3}Ai(-\nu^{2/3}t)\ dt\}$$

is an unbounded function of ν.

<u>Proof of (i)</u> On account of the regularity of the (C,k) process (Lemma 5.1), it will clearly be sufficient to prove (5.40) for the case $1/2 < k < 1$. We have

$$C_{k,\nu}\{\int_0^b f(t)\ \nu^{2/3}\ Ai(-\nu^{2/3}t)\ dt\}$$
$$=\int_0^b f(t)\ C_{k,\nu}\{\nu^{2/3}\ Ai(-\nu^{2/3}t)\}\ dt,$$

the interchange of order of integration being easily justified by Fubini's Theorem. The result (5.40) will now follow from Theorem 1.2 (i), provided we can show that, for $1/2 < k < 1$, the following three conditions hold:

$$(5.41)\qquad \lim_{\nu\to\infty}\int_0^\beta C_{k,\nu}\{\nu^{2/3}Ai(-\nu^{2/3}t)\}dt = 2/3,$$

for each β satisfying $0 < \beta \leq b$, and uniformly in β, $\epsilon \leq \beta \leq b$, for each ϵ satisfying $0 < \epsilon < b$;

$$(5.42)\qquad |C_{k,\nu}\{\nu^{2/3}\ Ai(-\nu^{2/3}t)\}| \leq K(\delta,k),\ \delta \leq t \leq b,\ \nu \geq \nu_o,$$

for some ν_o and for each δ satisfying $0 < \delta < b$; and

$$(5.43)\qquad \int_0^b |\ C_{k,\nu}\{\nu^{2/3}\ Ai(-\nu^{2/3}t)\}|\ dt \leq K(k),\ \nu \geq \nu_o.$$

For each β,(5.41) follows from (2.7) and the regularity of the (C,k) process. The uniformity assertion follows from Lemma 5.2. To see this, we notice that (5.41), with k = 0, clearly holds uniformly in β, $\epsilon \leq \beta \leq b$, and so the condition corresponding to hypothesis (5.9) of Lemma 5.2 is satisfied. We also see that

$$\int_0^\beta \nu^{2/3}\, Ai(-\nu^{2/3}t)\, dt = \int_0^{\nu^{2/3}\beta} Ai(-t)\, dt$$

and so $\int_0^\beta \nu^{2/3}\, Ai(-\nu^{2/3}t)\, dt$ is a bounded function of ν; thus,

$$\int_\delta^\gamma \left[\int_0^\beta \nu^{2/3}\, Ai(-\nu^{2/3}t)\, dt\right] d\nu \leq M(b),$$

for all subintervals $[\delta,\gamma]$ of $[0,b]$, and so the condition corresponding to hypothesis (5.10) of Lemma 5.2 is satisfied. Hence, the conclusion of Lemma 5.2 gives the uniformity in β of the relation (5.41).

The result (5.14) of Lemma 5.3 clearly shows that (5.42) holds, and (5.43) follows directly from the result (5.28) of Lemma 5.4. Thus (i) is proved.

Proof of (ii). Theorem 5.1 (ii) will follow from Theorem 1.2(ii) provided we can show that

$$\int_0^b \left|C_{1/2,\nu}\{\nu^{2/3}\, Ai(-\nu^{2/3}\, t)\}\right| dt$$

is an unbounded function of ν. It is clear from the result (5.27) of Lemma 5.4 that this is the case. This completes the proof of Theorem 5.1 (ii).

<u>Theorem 5.2</u>. (i) Let f be defined on $[0,b]$, $0 < b < \infty$, and suppose that

(5.44) $$f \in L[0,b];$$

(5.45) $$f(0+) \text{ exists};$$

(5.46) $$f(t) = f(0+) + O(t^{\alpha}),\ t \to 0+,$$

where α is some positive number. Then

(5.47) $$(C,k)\text{-}\lim_{\nu\to\infty}\int_0^b f(t)\ \nu^{2/3}\ \mathrm{Ai}(-\nu^{2/3}t)\ dt = \frac{2}{3}\ f(0+),\ k \geq 1/2.$$

(ii) For each $\alpha > 0$, and for each k, $0 \leq k < 1/2$, there exists a continuous f satisfying (5.46) such that

$$C_{k,\nu}\{\int_0^b f(t)\ \nu^{2/3}\ \mathrm{Ai}\ (-\nu^{2/3}t)\ dt\}$$

is an unbounded function of ν.

<u>Proof of (i)</u>. It will clearly be sufficient to prove that (5.47) holds in case $k = 1/2$. (The result for $k > 1/2$ then follows from the regularity of the (C,k) process, or directly from Theorem 5.1 (i)). To show that (5.47) holds in case $k = 1/2$, we proceed as follows:

We may assume without loss of generality that $\alpha < 1$; then

(5.48) $$C_{1/2,\nu}\{\int_0^b f(t)\ \nu^{2/3}\ \mathrm{Ai}(-\nu^{2/3}t)dt\}$$
$$= f(0+)\ C_{1/2,\nu}\{\int_0^b \nu^{2/3}\ \mathrm{Ai}(-\nu^{2/3}t)\ dt\}$$
$$+\int_0^b \frac{f(t)-f(0+)}{t^{\alpha}}\ C_{1/2,\nu}\{t^{\alpha}\nu^{2/3}\ \mathrm{Ai}(-\nu^{2/3}t)\}dt$$

where we have used the fact (easily justified by Fubini's Theorem) that:

$$C_{1/2,\nu}\left\{\int_0^b \frac{f(t) - f(0+)}{t^\alpha}\, t^\alpha \nu^{2/3}\, Ai(-\nu^{2/3}t)\, dt\right\}$$
$$= \int_0^b \frac{f(t) - f(0+)}{t^\alpha}\, C_{1/2,\nu}\{t^\alpha \nu^{2/3}\, Ai(-\nu^{2/3}t)\}\, dt.$$

The first expression on the right of (5.48) approaches $(2/3)f(0+)$, as $\nu \to \infty$, (by the regularity of the (C,k) process). It will follow from Theorem 1.2 (i) that the second expression on the right of (5.48) approaches 0 as $\nu \to \infty$, provided we can show that the following three conditions are satisfied: first,

$$\text{(5.49)} \qquad \lim_{\nu\to\infty}\int_0^\beta t^\alpha\, C_{1/2,\nu}\{\nu^{2/3}\, Ai(-\nu^{2/3}t)\}\, dt = 0,$$

for each β satisfying $0 < \beta \leq b$ and uniformly in β, $\epsilon \leq \beta \leq b$ for each ϵ satisfying $0 < \epsilon < b$; second,

$$\text{(5.50)} \qquad |t^\alpha\, C_{1/2,\nu}\{\nu^{2/3}\, Ai(-\nu^{2/3}t)\}| \leq K(\delta),\ \delta \leq t \leq b,\quad \nu \geq \nu_o,$$

for each δ satisfying $0 < \delta < b$, and some ν_o; and finally,

$$\text{(5.51)} \qquad \int_0^b t^\alpha |C_{1/2,\nu}\{\nu^{2/3}\, Ai(-\nu^{2/3}t)\}|\, dt \leq K,\ \nu \geq \nu_o.$$

It is clear from the regularity of the (C,k) process and Theorem 2.3 that (5.49) holds for each β concerned. We use Lemma 5.2 to prove the uniformity assertion. If $\epsilon \leq \beta \leq b$, we have

$$\int_\beta^b t^\alpha\, \nu^{2/3}\, Ai(-\nu^{2/3}t)\, dt = b^\alpha \int_{\xi(\nu)}^b \nu^{2/3}\, Ai(-\nu^{2/3}t)\, dt$$

where $\beta \leq \xi(\nu) \leq b$. The integral in the right member of this equality clearly approaches 0, uniformly in β, as $\nu \to \infty$.

Since we know that $\int_0^b t^\alpha\ \nu^{2/3}\ Ai(-\nu^{2/3}t)\ dt$ approaches 0,

uniformly in β (Theorem 2.2), we see that

$$\lim_{\nu\to\infty} \int_0^\beta t^\alpha\ \nu^{2/3}\ Ai(-\nu^{2/3}t)\ dt = 0 \tag{5.52}$$

uniformly in β, $\epsilon \leq \beta \leq b$, so that the condition corresponding to hypothesis (5.9) of Lemma 5.2 is satisfied. We also have

$$\left|\int_0^\beta t^\alpha\ \nu^{2/3}\ Ai(-\nu^{2/3}t)dt\right| \leq K\nu^{2/3}$$

where K is independent of ν, and it follows that

$$\int_\gamma^\delta \left[\int_0^\beta t^\alpha\ \nu^{2/3}\ Ai(-\nu^{2/3}\ t)\ dt\right] d\nu \leq M(b),$$

for all subintervals $[\gamma,\delta]$ of $[0,b]$; hence, the condition corresponding to hypothesis (5.10) of Lemma 5.2 is satisfied. The conclusion of Lemma 5.2 now implies that the uniformity assertion in (5.49) holds.

The result (5.14) of Lemma 5.3 shows that (5.50) is satisfied, while (5.51) follows from the result (5.28) of Lemma 5.4.

This completes the proof of Theorem 5.2 (i).

<u>Proof of (ii)</u>. The unboundedness in ν of

$$\int_0^b t^\alpha |C_{k,\nu}\{\nu^{2/3}\ Ai(-\nu^{2/3}t)\}|\ dt$$

(Lemma 5.4, (5.25)) shows, by Theorem 1.2 (ii), that there exists a continuous function g on [0,b] for which

$$\int_0^b g(t)t^\alpha\ C_{k,\nu}\{\nu^{2/3}\ Ai(-\nu^{2/3}t)\}\ dt$$

is an unbounded function of ν. If we choose $f(t) = g(t)t^\alpha$,

$0 \leq t \leq b$, then f clearly satisfies all the conditions in the statement of the theorem, and the proof of Theorem 5.2 (ii) is complete.

Theorem 5.3 Let f be defined on $[0,b]$, $0 < b < \infty$, and suppose that:

(5.53) $$f \in L(0,b);$$

(5.54) $$f(0+) \text{ exists};$$

(5.55) $$C_{k,\nu}\{\int_0^b f(t)\ w(\nu,-t)\ dt\} \text{ exists for each}$$

$\nu \geq \nu_o$ (some ν_o), and each $k > 1/2$, where $w(\nu,t)$ is the solution of (3.4) which we introduced in Chapter 3. (We are now writing $b = -a$, in the notation of Chapter 3).

Then, for $k > 1/2$,

(5.56) $$(C,k)\text{-}\lim_{\nu\to\infty} \int_0^b f(t)\ w(\nu,-t)\ dt = \frac{2}{3} f(0+).$$

If, in addition f satisfies

(5.57) $$f(t) = f(0+) + O(t^{\alpha}),\ t \to 0+,\ \alpha > 0,$$

and if, for all sufficiently large ν,

$$C_{1/2,\nu}\{\int_0^b f(t)\ w(\nu,-t)\ dt\}$$

exists, then (5.56) holds in the case $k = 1/2$.

Proof We use the approximation (3.5),

$$w(\nu,-t) = \nu^{2/3} Ai(-\nu^{2/3}t) + \epsilon(\nu,-t).$$

It is clear from Theorem 5.1 (i) that the contribution from the principal term here to the limit in (5.56) is exactly (2/3) f(0+). The same result follows from Theorem 5.2 (i) in the case $k = 1/2$. We have, from (3.7),

$$\int_0^b f(t)\,\epsilon\,(\nu,-t)\;dt \;=\; O(\nu^{-1/3})\int_0^b |f(t)|dt$$

so that

$$\lim_{\nu\to\infty}\int_0^b f(t)\;\epsilon\;(\nu,-t)\;dt = 0.$$

Hence, for $k > 0$,

$$(C,k)\text{-}\lim_{\nu\to\infty}\int_0^b f(t)\;\epsilon\;(\nu,-t)\;dt = 0,$$

and so the contribution from $\epsilon(\nu,-t)$ to the limit in (5.56) is zero for all cases considered. This completes the proof of Theorem 5.3.

CHAPTER VI

THE GIBBS PHENOMENON

We begin by giving the definition of the Gibbs phenomenon which we will use. R. L. Forbes* gives a discussion of the relationship of various such definitions occurring in [36], [37] and elsewhere. A discussion of various situations in which the phenomenon occurs is given in F. Ustina's thesis [33].

Definition 6.1. Let $\Phi(\nu,x)$ be defined for $\nu \geq \nu_o$ and for $-\delta \leq x \leq \delta$, and suppose that

(6.1) $$\lim_{\nu\to\infty} \Phi(\nu,x) = f(x), \quad -\delta \leq x \leq \delta.$$

We define the Gibbs set of $\Phi(\nu,x)$, at $x = 0$, as the aggregate of all limit points, as $\nu \to \infty$, of functions $\Phi(\nu,x_\nu)$, where $x_\nu \to 0$ as $\nu \to \infty$. We say that $\Phi(\nu,x)$ exhibits the Gibbs phenomenon at $x = 0$ if the Gibbs set of $\Phi(\nu,x)$ at $x = 0$ contains points outside the closed interval

(6.2) $$[\liminf_{x\to 0} f(x), \limsup_{x\to 0} f(x)].$$

We will show in this chapter that for a certain class of functions f, having a simple discontinuity at $x = 0$

(6.3) $$C_{k,\nu}\{\int_a^\infty f(t)\ \nu^{2/3}\ Ai\ [\nu^{2/3}(t-x)]dt\}$$

exhibits the Gibbs phenomenon at $x = 0$ if, and only if,

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*The Gibbs Phenomenon of Sequences and Fourier Series, M.Sc. Thesis, University of Alberta, 1962.

$$\max_{x>0} \frac{1}{2}\int_0^x (1-\frac{t}{x})^k \, [J_{1/3}(t) + J_{-1/3}(t)] \, dt > 1. \tag{6.4}$$

Using (6.4), we can show that there is a Gibbs phenomenon for $0 \le k \le 1$ and numerical work suggests that there is no Gibbs phenomenon for $k \geq 2$. Lorch and Szego ([20(a)], Theorem 2) show that there is a Gibbs phenomenon, for ordinary convergence, in the case of the kernel $\nu J_\nu(\nu t)$ obtaining the condition (6.4) above, with $k = 0$.

We show that, for the same class of functions f,

$$C_{k,\nu}\{\int_a^\infty f(t)\nu \, Ai(\nu,t-x) \, dt\}$$

exhibits the Gibbs phenomenon at $x = 0$ if, and only if,

$$\max_{x>0} \int_0^x (1 - \frac{t}{x})^k \, Ai(-t) \, dt > \frac{2}{3}. \tag{6.5}$$

Using condition (6.5), we show that there is a Gibbs phenomenon in the case $k = 0$, but none for $k \geq 2$. Numerical work suggests that in this case there is no Gibbs phenomenon for $k \geq 1$.

The type of problem considered here, had its origin in Fourier series. H. Cramér [4] (using a definition of (C,k) - summability appropriate to summation of series) showed that the (C,k) means of the Fourier series of a certain class of functions (with simple discontinuity) exhibit the Gibbs phenomenon if, and only if,

$$\max_{x>0} \int_0^x (1-\frac{t}{x})^k \frac{\operatorname{Sin} t}{t} \, dt > \frac{\pi}{2}. \tag{6.6}$$

Cramér showed that in this case there exists a k_o, $0 < k_o < 1$, such that there is a Gibbs phenomenon for $0 \le k < k_o$, but

none for $k \geq k_o$. T. H. Gronwall [13] showed that $k_o = 0.43955\ldots.$

A condition somewhat similar to our conditions (6.4) and (6.5) occurs in some work of C. N. Moore [23], on Gibbs phenomenon for developments in Fourier-Bessel series of a certain type of function. The condition is

$$\max_{x>0} \int_0^x \left(1 - \frac{t}{x}\right)^k \nu \frac{J_\nu(t)}{t}\, dt > 1. \tag{6.7}$$

Moore showed that there is a number r between 0 and 1, such that (6.7) holds for $0 \leq k < r$ but not for $k \geq r$.

We will use Theorem 6.1, below, to derive the conditions (6.4) and (6.5). The proof of Theorem 6.1 will depend on several lemmas.

<u>Lemma 6.1</u> Let $-\infty < a < -\delta < 0$, let f be defined on $[a,\infty)$, and suppose that:

$$f \in L(-\delta,\infty); \tag{6.8}$$

$$f \in BV[a,\delta]; \tag{6.9}$$

$$f \in C[-\delta,0],\ f \in C[0,\delta], \tag{6.10}$$

$$\lim_{h\to 0+} f(-\delta-h) = f(-\delta), \text{ and } \lim_{h\to 0+} f(\delta+h) = f(\delta).$$

Let g(t) be defined by

$$\begin{aligned} g(t) &= f(t) - f(0-), \quad a \leq x < 0, \\ g(t) &= f(t) - f(0+), \quad 0 \leq x < \infty, \end{aligned} \tag{6.11}$$

and let $\Phi(\nu,x)$ be defined by

$$\Phi(\nu,x) = \int_a^\infty f(t)\, \nu \operatorname{Ai}[\nu(t-x)]\, dt, \quad \nu > 0, \quad -\delta \leq x \leq \delta. \tag{6.12}$$

Then:

(6.13) $$\Phi(\nu,x) = f(0-)+[f(0+)-f(0-)]\left[\tfrac{1}{3}+\int_0^{x\nu} Ai(-\theta)\,d\theta\right] + g(x) + \eta(\nu,x),$$

where

(6.14) $$\eta(\nu,x) = o(1), \text{ uniformly in } x,\ -\delta \le x \le \delta,$$

as $\nu \to \infty$, and

(6.15) $$|\eta(\nu,x)| \le M(b), \text{ for } -\delta \le x \le \delta, \text{ and } 0 \le \nu \le b.$$

Proof. We first point out that (6.15) follows from the continuity of $g(x)$, the boundedness in ν and x of $\int_0^{x\nu} Ai(-\theta)\,d\theta$, and the fact that $|\Phi(\nu,x)| \le \nu\int_a^\infty |f(t)|dt$.

We have

(6.16) $$\Phi(\nu,x) = f(0-)\int_a^\infty \nu Ai[\nu(t-x)]\,dt + [f(0+) - f(0-)]\left[\int_x^\infty \nu Ai[\nu(t-x)]\,dt + \int_0^x \nu Ai[\nu(t-x)]\,dt\right] + \int_a^\infty g(t)\,\nu\,Ai[\nu(t-x)]\,dt.$$

Now, it is clear from Theorem 2.4 that

(6.17) $$\int_a^\infty \nu Ai[\nu(t-x)]\,dt = 1 + o(1)$$

uniformly in x, $-\delta \le x \le \delta$, and

(6.18) $$\int_a^\infty g(t)\nu Ai[\nu(t-x)]\,dt = g(x) + o(1),$$

uniformly in x, $-\delta \le x \le \delta$. We also have, from (2.7)

(6.19) $$\int_x^\infty \nu Ai[\nu(t-x)]\,dt = 1/3,$$

and

(6.20) $$\int_0^x \nu Ai[\nu(t-x)]\,dt = \int_0^{x\nu} Ai(-\theta)\,d\theta.$$

If we substitute (6.17) to (6.20) into (6.16) we

immediately get (6.13), where $\eta(\nu,x)$ satisfies (6.14). This completes the proof of Lemma 6.1.

Lemma 6.2. If $\rho > 0$ and $\Phi(\nu,x)$ is given by (6.12), then

(6.21) $$C_{k,\nu}\{\Phi(\nu^\rho,x)\} = f(0-) + [f(0+) - f(0-)][\tfrac{1}{3} + G(x\nu^\rho,\rho,k)] + g(x) + o(1),$$

uniformly in x, $-\delta \leq x \leq \delta$, as $\nu \to \infty$, where

(6.22) $$G(\sigma,\rho,k) = \int_0^\sigma [1-(\tfrac{t}{\sigma})^{1/\rho}]^k \operatorname{Ai}(-t)\,dt, \quad \sigma \neq 0,$$
$$G(0,\rho,k) = 0,$$

for $\rho > 0$, $k \geq 0$.

Proof. We use the result (6.13) of Lemma 6.1. We see from Lemma 5.2, on account of (6.14) and (6.15), that

(6.23) $$C_{k,\nu}\{\eta(\nu^\rho,x)\} = o(1), \quad \nu \to \infty,$$

uniformly in x, $-\delta \leq x \leq \delta$. Thus (6.21) will be proved if we can show that

(6.24) $$C_{k,\nu}\{\int_0^{x\nu^\rho} \operatorname{Ai}(-\theta)\,d\theta\} = G(x\nu^\rho,\rho,k)$$

where $G(\sigma,\rho,k)$ is given by (6.22). This is clearly true for $k = 0$. It is also obvious for $k > 0$, $x = 0$. Now, for $k > 0$, $x \neq 0$,

$$C_{k,\nu}\{\int_0^{x\nu^\rho} \operatorname{Ai}(-\theta)\,d\theta\}$$

$$= k\nu^{-k} \int_0^\nu (\nu-p)^{k-1} [\int_0^{xp^\rho} \operatorname{Ai}(-\theta)\,d\theta]\,dp$$

$$= \int_0^{x\nu^\rho} [1-(\frac{\theta}{x\nu^\rho})^{1/\rho}]^k \operatorname{Ai}(-\theta)\,d\theta = G(x\nu^\rho,\rho,k),$$

the interchange in the order of integrations being easily justified. Thus, (6.21) holds, and so the proof of Lemma 6.2 is complete.

Lemma 6.3 Let $G(\sigma,\rho,k)$ be defined as in (6.22). Then for each $k \geq 0$, and each $\rho > 0$:

(i) $G(\sigma,\rho,k)$ increases from $-1/3$ to 0 as σ increases from $-\infty$ to 0;

(ii) $G(\sigma,\rho,k) \geq 0$ for $\sigma > 0$; and

(iii) $\lim\limits_{\sigma\to\infty} G(\sigma,\rho,k) = \frac{2}{3}$.

Proof. We have

$$(6.25)\quad \frac{d}{d\sigma} G(\sigma,\rho,k) = \frac{k}{\rho\sigma^2}\int_0^{\sigma} t\left(\frac{t}{\sigma}\right)^{1/\rho-1}\left[1-\left(\frac{t}{\sigma}\right)^{1/\rho}\right]^{k-1} \mathrm{Ai}(-t)\, dt$$

and this is positive for $\sigma < 0$, because then the integrand is negative. Thus $G(\sigma,\rho,k)$ is increasing as σ increases from $-\infty$ to 0. We may use the second mean-value theorem to show that $G(\sigma,\rho,k)$ is a continuous function of σ at $\sigma = 0$. Thus, $G(\sigma,\rho,k)$ increases to 0 as σ increases to 0. We have

$$(6.26)\qquad G(\sigma,\rho,k) = \int_0^{-\sigma}\left[1-\left(\frac{\theta}{-\sigma}\right)^{1/\rho}\right]^k \mathrm{Ai}(\theta)\, d\theta$$

$$= \int_0^{(-\sigma)^{1/\rho}}\left[1-\frac{\phi}{(-\sigma)^{1/\rho}}\right]^k \mathrm{Ai}(\phi^{\rho})\rho\,\phi^{\rho-1}\, d\phi$$

Now, by the regularity of (C,k) summability of integrals (Hardy [14], p. 111) we see that

$$\lim_{\sigma\to-\infty}\int_0^{(-\sigma)^{1/\rho}}\left[1-\frac{\phi}{(-\sigma)^{1/\rho}}\right]^k \mathrm{Ai}(\phi^{\rho})\,\rho\,\phi^{\rho-1} d\phi$$

exists and is equal to

$$\lim_{\sigma\to-\infty} \int_0^{(-\sigma)^{1/\rho}} Ai(\phi^\rho)\, \rho\phi^{\rho-1}\, d\phi$$

$$= -\int_{-\infty}^{0} Ai(\theta)\, d\theta = -1/3.$$

This completes the proof of (i).

Now, $[1 - (t/\sigma)^{1/\rho}]^k$ is a decreasing function of t, for $0 \leq t \leq \sigma$. Hence, the second mean-value theorem gives

(6.27) $$G(\sigma,\rho,k) = \int_0^{\xi(\rho,k)} Ai(-t)\, dt, \quad 0 \leq \xi(\rho,k) \leq \nu.$$

Now, as we remarked in Chapter 2, when $t > 0$, $Ai(-t)$ is an oscillating function whose successive arches bound decreasing areas, the area in the first arch being positive. Thus

$$\int_0^{\xi(\rho,k)} Ai(-t)\, dt \geq 0,$$

and so (ii) is proven.

The proof of (iii) is very similar to that of (6.26), so it will not be given here.

Lemma 6.4 Let $G(\sigma,\rho,k)$ be given by (6.22). Then, for each $\rho > 0$,

$$\max_{\sigma>0} G(\sigma,\rho,k)$$

is a nonincreasing function of k, $0 \leq k < \infty$.

Proof. We suppose that $k \geq 0$, $\alpha > 0$, and that

(6.28) $$\max_{\sigma>0} G(\sigma,\rho,k) = M.$$

We wish to show that

(6.29) $$\max_{\sigma>0} G(\sigma,\rho,k+\alpha) \leq M.$$

We have:

$$G(\sigma, \rho, k+\alpha) = \int_0^\sigma [1 - (t/\sigma)^{1/\rho}]^{k+\alpha} Ai(-t)\, dt$$

$$= \tau^{-k-\alpha} \int_0^\tau (\tau-\phi)^{k+\alpha} Ai(-\phi^\rho)\, \rho\phi^{\rho-1}\, d\phi, \quad (\tau = \sigma^{1/\rho})$$

$$= \frac{\tau^{-k-\alpha}}{B(\alpha,k+1)} \int_0^\tau (\tau-\phi)^{\alpha-1} \left[\int_0^\phi (\phi-q)^k Ai(-q^\rho)\rho q^{\rho-1} dq\right] d\phi$$

$$= \frac{\tau^{-k-\alpha}}{B(\alpha,k+1)} \int_0^\tau (\tau-\phi)^{\alpha-1}\, \phi^k\, G(\phi^\rho, \rho, k)\, d\phi;$$

the second equality follows from (5.8). Thus, by (6.28) we have

$$|G(\sigma,\rho,k+\alpha)| \le \frac{\tau^{-k-\alpha}}{B(\alpha,k+1)} M \int_0^\tau (\tau-\phi)^{\alpha-1} \phi^k d\phi = M.$$

Hence (6.29) holds, and the proof of Lemma 6.4 is complete.

Theorem 6.1. Let $\Phi(\nu,x)$ be defined as in Lemma 6.1 above, i.e.,

(6.30) $$\Phi(\nu,x) = \int_a^\infty f(t)\, \nu\, Ai[\nu(t-x)]dt,$$

for $\nu > 0$, $-\delta \le x \le \delta$, where f is defined on $[a,\infty)$, $-\infty < a < -\delta < 0$, and

(6.31) $f \in L(-\delta,\infty)$;

(6.32) $f \in BV[a,\delta]$;

(6.33) $f \in C[-\delta,0]$, $f \in C[0,\delta]$.

Then for $\rho > 0$, the Gibbs set for $C_{k,\nu}\{\Phi(\nu^\rho,x)\}$ at $x = 0$ is the closed interval whose endpoints are $f(0-)$ and $f(0-) + [f(0+) - f(0-)][\frac{1}{3} + \max_{\sigma>0} G(\sigma,\rho,k)]$.

Remark. It is clear from this theorem and the definition of the Gibbs phenomenon that, for fixed ρ and k, $C_{k,\nu}\{\Phi(\nu^\rho,x)\}$ exhibits the Gibbs phenomenon at $x = 0$, if and only if

$$G(\sigma,\rho,k) > 2/3,$$

for some $\sigma > 0$.

Proof. It is clear from (6.33) that $f(0+)$ and $f(0-)$ exist. We suppose, first, that

$$f(0+) \geq f(0-).$$

Then:

(6.34) $$\limsup_{x\to 0} f(x) = f(0+),$$

(6.35) $$\liminf_{x\to 0} f(x) = f(0-).$$

Let x_ν be a function of ν which tends to 0 as ν approaches infinity. Then, by Lemma 6.2, we have

(6.36) $$\limsup_{\nu\to\infty} C_{k,\nu}\{\Phi(\nu^\rho, x_\nu)\} = f(0-) + [f(0+) - f(0-)][\tfrac{1}{3} + \limsup_{\nu\to\infty} G(x_\nu\nu^\rho,\rho,k)]$$

and

(6.37) $$\liminf_{\nu\to\infty} C_{k,\nu}\{\Phi(\nu^\rho, x_\nu)\} = f(0-) + [f(0+) - f(0-)][\tfrac{1}{3} + \liminf_{\nu\to\infty} G(x_\nu\nu^\rho,\rho,k)]$$

Now, we see from Lemma 6.3 that $\limsup_{\nu\to\infty} G(x_\nu\nu^\rho,\rho,k)$ and $\liminf_{\nu\to\infty} G(x_\nu\nu^\rho,\rho,k)$ must both lie between $-1/3$ and $\max_{\sigma>0} G(\sigma,\rho,k)$. Moreover, if we choose $x_\nu = \sigma\nu^{-\rho}$, where $-\infty < \sigma < \infty$, we see that

$$\lim_{\nu\to\infty} G(x_\nu\nu^\rho,\rho,k) = G(\sigma,\rho,k)$$

Since $G(\sigma,\rho,k)$ is a continuous function of σ we thus see that the set of limit points of

$$G(x_\nu\nu^\rho,\rho,k)$$

for all $x_\nu \to 0$ is the interval

(6.38) $$[-1/3, \max_{\sigma>0} G(\sigma,\rho,k)].$$

On using (6.36) and (6.37) we see that the set of limit points of

$$C_{k,\nu}\{\Phi(\nu^{\rho}, x_{\nu})\}$$

is precisely the interval described in the statement of the theorem.

Thus, Theorem 6.1 is proven in the case $f(0+) \geq f(0-)$, and only minor changes are required in the case $f(0+) < f(0-)$.

Corollary 6.1 The function

$$C_{k,\nu}\{\Phi(\nu, x)\}$$

exhibits the Gibbs phenomenon at $x = 0$, when $k = 0$ but not when $k \geq 2$.

Proof: It will be sufficient to show that

$$\max_{\sigma>0} G(\sigma,1,0) > 2/3 \tag{6.39}$$

and

$$\max_{\sigma>0} G(\sigma,1,2) \leq 2/3 \tag{6.40}$$

because, by Lemma 6.4, (6.40) will imply

$$\max_{\sigma>0} G(\sigma,1,k) \leq 2/3, \quad k > 2.$$

Now,

$$G(\sigma,1,0) = \int_0^{\sigma} Ai(-t)\, dt,$$

and this can be as large as $(2/3)(1.4115\ldots)$, if σ is chosen to be the first positive zero of $Ai(-t)$, as Lorch and Szego point out ([21], p.210). Thus, (6.39) holds.

We also have

$$\frac{d}{d\sigma} G(\sigma,1,2) = \frac{2}{\sigma^2}\int_0^{\sigma} (1-\frac{t}{\sigma})\, t\, Ai(-t)\, dt$$

$$= -\frac{2}{\sigma^2}\int_0^{\sigma} (1-\frac{t}{\sigma})\, \frac{d^2}{dt^2}[Ai(-t)]\, dt,$$

on using the differential equation satisfied by the Airy function. Integration by parts gives

(6.41) $$\frac{d}{d\sigma} G(\sigma,1,2) = -\frac{2}{\sigma^3} [\sigma Ai'(0) + Ai(-\sigma) - Ai(0)].$$

If we use (2.3) and ([34], p. 49 (1))

(6.42) $$J_\nu(x) \leq \frac{(x/2)^\nu}{\Gamma(\nu+1)}, \quad \nu > -1/2, \; x > 0,$$

we get

(6.43) $$Ai(-\sigma) \leq Ai(0) - \sigma Ai'(0), \quad \sigma > 0,$$

and hence, by (6.41), $G(\sigma,1,2)$ is an increasing function of σ, $0 < \sigma < \infty$. Since by Lemma 6.3 (iii), $G(\sigma,1,2)$ approaches $2/3$ as $\sigma \to \infty$, we see that (6.40) holds.

Corollary 6.2 The function

$$C_{k,\nu}\{\Phi(\nu^{2/3},x)\}$$

exhibits the Gibbs phenomenon at $x = 0$, when $0 \leq k \leq 1$.

Proof. In view of Lemma 6.4, it will be sufficient to show that

(6.44) $$\max_{\sigma>0} G(\sigma,2/3,1) > 2/3.$$

We have

$$G(\sigma,\tfrac{2}{3},1) = \int_0^\sigma [1-(\tfrac{t}{\sigma})^{3/2}] \, Ai(-t) \, dt = \frac{1}{3}\int_0^\tau (1-\tfrac{t}{\tau})[J_{1/3}(t) + J_{-1/3}(t)] \, dt$$

by (2.2), where $\tau = (2/3)\sigma^{3/2}$.

If we use the series expansion for $J_{1/3}(t) + J_{-1/3}(t)$, we find that

$$G(6^{2/3},\tfrac{2}{3},1) = (1.079\ldots)2/3$$

and so (6.44) is satisfied.

This completes the proof of Corollary 6.2.

CONCLUDING REMARKS

1. Theorem 2.3 shows that our results for the kernel $\nu Ai(\nu t)$ may be extended to the case in which the interval of integration is the whole real line. We have not made this extension in the case of the kernel $w(\nu,t)$ but consider merely the interval $[a,\infty)$, where $-\infty < a < 0$. The reason for this is that the bound for the error term, $\epsilon(\nu,t)$, in the oscillatory region of $w(\nu,t)$ is non-negative (Appendix (A.17)), and hence the desirable oscillatory property of the main term is not present in this bound for the error term. Thus, the results which would be found by using this approximation in the infinite case are not as strong as those found for the Airy function kernel. It may be possible to obtain better results by using some other type of asymptotic approximation for solutions in an oscillatory region.

2. The solution $w(\nu,t)$ of

$$d^2w/dt^2 = \{\nu^2 t + q(t)\}w$$

which we have used as a kernel in Chapter 3 is a "principal solution at $t = \infty$" in the sense of W. Leighton and M. Morse (Singular Quadratic Functionals, Trans. Amer. Math. Soc., vol. 40, 1936, pp. 252-286). In [15], p. 355, the concept "principal solution" is defined and it is shown that principal solutions are determined uniquely up to a constant multiplicative factor. It does not appear that useful results on convergence of singular integrals can be obtained by

considering non-principal solutions of the above differential equation. Lorch and Szego showed that their results for the convergence of singular integrals with the kernel $\nu J_\nu(\nu t)$ can not be generalized by replacing $J_\nu(\nu t)$ by $\mathcal{B}_\nu(\nu t)$, where $\mathcal{B}_\nu$ is a general solution of Bessel's differential equation. In fact, they showed ([21], pp. 215-217):

$$\lim_{\nu\to\infty} \int_1^A \nu\, Y_\nu(\nu t)dt = 0, \ A > 1,$$

$$\lim_{\nu\to\infty} \int_b^1 \nu \quad Y_\nu(\nu t)\ dt = -\infty, \ 0 \le b < 1.$$

We find similarly, in the case of the Airy function Bi(t), that with $b > 0$, $a < 0$,

$$\lim_{\nu\to\infty} \int_0^b \nu \mathrm{Bi}(\nu t)\ dt = \int_0^\infty \mathrm{Bi}(t)dt = \infty, \text{ and}$$

$$\lim_{\nu\to\infty} \int_a^0 \nu \mathrm{Bi}(\nu t)\ dt = \int_{-\infty}^0 \mathrm{Bi}(t)\ dt = 0,$$

the values of these integrals being given in [1], p. 450, 10.8.84 and 10.8.85. It may be possible to obtain useful results by considering integrals in which the functions $\nu Y_\nu(\nu t)$ and $\nu \mathrm{Bi}(\nu t)$ are multiplied by suitable functions of ν and t.

3. In the case of the Airy function kernel we were able to show that

$$\lim_{\nu\to\infty} \int_0^\infty f(t) \quad \nu \mathrm{Ai}(\nu t)\ dt = \tfrac{1}{3}f(0),$$

when f satisfies an integrability condition and, in addition,

$$\lim_{h\to 0+} h^{-1}\int_0^h f(t)\ dt = f(0).$$

It would be of interest to see whether this last condition could be replaced by:

$$\lim_{h\to 0+} kh^{-k}\int_0^h (h-t)^{k-1} f(t)\,dt = f(0),\ k > 1.$$

4. Several questions may be posed with regard to "degree of convergence" of the singular integrals which we consider. Thus, we might ask whether there exists an α, such that

$$\int_a^\infty f(t)\,w(\nu,t-x)\,dt - f(x) = O(\nu^{-\alpha})$$

uniformly in an x-interval, as $\nu \to \infty$, for all f belonging to (say) the class of differentiable functions.

5. We showed in Chapter 5 that the "Lebesgue constants" $\Lambda_k(\nu,0)$ were unbounded functions of ν for $0 \le k \le 1/2$ and bounded for $k > 1/2$. For each fixed k, we might ask whether $\Lambda_k(\nu,0)$ is a monotonic function of ν, or even whether its higher derivatives (with respect to ν) are monotonic functions of ν.

6. We may define (C,k) summability for $-1 < k < 0$ (cf. [14], p. 111). It may be of interest to see whether our various convergence theorems still hold with this type of summability and also to discuss the Gibbs phenomenon for these cases.

7. The connection which we have established between differential equations with simple transition points, and singular integrals, may have extensions to other types of transition points and singularities; e.g., we might consider solutions of

$$d^2w/dt^2 = \{\nu^2 t^n + q(t)\}w, \quad n > 1.$$

It would also be interesting to see whether the type of procedure which we use in Chapters 3 and 4 might be applied to singular integrals arising from various orthogonal expansions.

8. Some functions defined by contour integrals have asymptotic expansions involving Airy functions; see, e.g., [9] and the references given there. This suggests a way, other than the one we have used, in which the Airy function kernel of Chapter 2 might be generalized.

9. With regard to Remark 4, above, on "degree of convergence" some further work of Olver [27] may be useful since it gives error bounds for higher order approximations to $w(\nu,t)$ in terms of Airy functions.

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APPENDIX

PROOF OF THEOREM 3.1

The standard solutions Ai(x), Bi(x) of

$$\text{(A.1)} \qquad \frac{d^2y}{dx^2} = xy$$

are defined by the initial conditions:

$$\text{(A.2)} \qquad Ai(0) = \frac{1}{\sqrt{3}}\, Bi(0) = \frac{1}{3^{2/3}\Gamma(2/3)},$$

$$\text{(A.3)} \qquad Ai'(0) = -\frac{1}{\sqrt{3}}\, Bi'(0) = -\frac{1}{3^{1/3}\,\Gamma(1/3)},$$

and satisfy the Wronskian relation:

$$\text{(A.4)} \qquad Ai(x)\, Bi'(x) - Ai'(x)\, Bi(x) = 1/\pi.$$

Ai(x), Bi(x) are oscillatory for $x < 0$,

and are positive and monotonic for $x > 0$,

their asymptotic behaviour being given by:

$$\text{(A.5)} \qquad Ai(x) = \tfrac{1}{2}\pi^{-\frac{1}{2}}x^{-\frac{1}{4}}[\exp(-\tfrac{2}{3}\, x^{3/2})][1 + O(x^{-3/2})],\ x \to \infty,$$

$$\text{(A.6)} \qquad Bi(x) = \pi^{-\frac{1}{2}}x^{-\frac{1}{4}}[\exp(\tfrac{2}{3}\, x^{3/2})][1 + O(x^{-3/2})],\ x \to \infty.$$

As in [26], we introduce the notation

$$E(x) = \exp(\tfrac{2}{3}\, x^{3/2}),\ x > 0,$$

$$\text{(A.7)} \qquad E(x) = 1,\ x \le 0,$$

$$E^{-1}(x) = 1/E(x),$$

and define four auxiliary functions, $M(x)$, $\chi(x)$, $N(x)$, $\psi(x)$ by the equations:

$$\begin{aligned} Ai(x) &= E^{-1}(x)\ M(x)\ \sin\ \chi(x), \\ Bi(x) &= E(x)\ M(x)\ \cos\ \chi(x), \\ Ai'(x) &= E^{-1}(x)\ N(x)\ \sin\ \psi(x), \\ Bi'(x) &= E(x)\ N(x)\ \cos\ \psi(x). \end{aligned} \tag{A.8}$$

Thus

$$M^2(x) = E^2(x)\ Ai^2(x) + E^{-2}(x)\ Bi^2(x). \tag{A.9}$$

From (A.5) and (A.6) we find:

$$M(x) \cong (5/4\pi)^{\frac{1}{2}}\ x^{-\frac{1}{4}},\quad x \to \infty. \tag{A.10}$$

It is clear from (A.9) and (A.10) that if $-\infty < a < 0$,

$$\lambda = \max_{(a,\infty)} \{\pi |x|^{\frac{1}{2}}\ M^2(x)\} \tag{A.11}$$

exists, the value of λ being unimportant, from our point of view.

Using (A.8) we get as in [26], p. 754,

$$|Ai(x)\ Bi(t) - Bi(x)\ Ai(t)| \le E^{-1}(x)E(t)M(x)M(t),\quad t \ge x, \tag{A.12}$$

and

$$|Ai'(x)Bi(t)-Bi'(x)Ai(t)| \le E^{-1}(x)E(t)N(x)M(t),\quad t \ge x. \tag{A.13}$$

Theorem A. 1

Let $f(x)$ be a continuous function of x on the interval $a \le x < \infty$, where $-\infty < a < 0$, and let

$$F(x) = \int_x^{\infty} |t^{-\frac{1}{2}} f(t)|\ dt \tag{A.14}$$

exist for $a \le x < \infty$. Then the differential equation

$$d^2w/dx^2 = \{x + f(x)\}w \tag{A.15}$$

has a solution

(A.16) $$w(x) = Ai(x) + \epsilon(x),$$

where

(A.17) $$|\epsilon(x)| \le E^{-1}(x)M(x)(e^{\lambda F(x)}-1),\ a \le x < \infty,$$

E, M, and λ being given by (A.7), (A.9), and (A.11) respectively. Moreover, the condition

(A.18) $$\lim_{x\to\infty} x^{\frac{1}{4}}\, E(x)\, w(x) = \tfrac{1}{2}\pi^{-\frac{1}{2}}$$

determines the solution $w(x)$ of (A.15) uniquely.

<u>Proof</u>

We introduce a sequence of functions

$$h_0(x) = 0,$$

(A.19) $$h_n(x) = \pi\int_x^\infty [Ai(x)Bi(t) - Bi(x)Ai(t)]f(t)\,[h_{n-1}(t) + Ai(t)]\,dt,\ n \ge 1.$$

In case $n = 1$, it follows from (A.12) and (A.8) that the integrand here is bounded by

$$E^{-1}(x)\ M(x)\ M^2(t)|f(t)|.$$

Thus, on account of (A.14) and (A.10), the integral defining $h_1 - h_0$ converges and, by (A.11),

(A.20) $$|h_1(x) - h_0(x)| \le E^{-1}(x)M(x)\lambda F(x),\ a \le x < \infty.$$

Suppose now that the integral defining $h_n(x) - h_{n-1}(x)$ converges and that

(A.21) $$|h_n(x) - h_{n-1}(x)| \le E^{-1}(x)M(x)\ \lambda^n\ (F(x))^n/n!.$$

The integral defining $h_{n+1}(x) - h_n(x)$ is

$$\pi\int_x^\infty [Ai(x)Bi(t) - Bi(x)Ai(t)]f(t)[h_n(t)-h_{n-1}(t)]dt.$$

Now, on account of (A.12) and (A.21) the integrand here is bounded by

$$E^{-1}(x)M(x) \quad \pi M^2(t)|f(t)| \quad \frac{\lambda^n(F(t))^n}{n!}$$

$$= -E^{-1}(x)M(x) \quad \pi|t|^{\frac{1}{2}}M^2(t) \quad \frac{\lambda^n(F(t))^n}{n!} \frac{d(F(t))}{dt} .$$

Thus the integral defining $h_{n+1}(x) - h_n(x)$ converges and

(A.22) $$h_{n+1}(x)-h_n(x) \leq E^{-1}(x)M(x) \quad \frac{\lambda^{n+1}(F(x))^{n+1}}{(n+1)!} .$$

Since this holds for $n = 0$ ((A.20)), we see that (A.22) is true for $n = 0, 1, 2, \ldots$.

The inequalities (A.22) show that the series

$$\sum_{n=0}^{\infty} \{h_{n+1}(x) - h_n(x)\}$$

converges uniformly to a function $\epsilon(x)$ for $a \leq x < \infty$, and

$$|\epsilon(x)| \leq E^{-1}(x)M(x)(e^{\lambda F(x)}-1), \quad a \leq x < \infty .$$

This is just (A.17). We must now show that with

(A.23) $$\epsilon(x) = \sum_{n=0}^{\infty} \{h_{n+1}(x) - h_n(x)\},$$

$w(x) = Ai(x) + \epsilon(x)$ satisfies (A.15). We will first need to show that the series in (A.23) may be differentiated twice to yield $\epsilon''(x)$.

It may be differentiated once to yield $\epsilon'(x)$ provided that the differentiated series is uniformly convergent ([32], § 1.72).

We have

$$\text{(A.24)} \qquad h_1'(x) = \pi\int_x^\infty [Ai'(x)Bi(t)-Bi'(x)Ai(t)]f(t)Ai(t)\,dt.$$

(A.13) and (A.8) show that this integral is convergent and give:

$$\text{(A.25)} \qquad |h_1'(x)| \le \lambda E^{-1}(x)N(x)F(x),\ a \le x < \infty$$

We can proceed in this way to show that

$$\text{(A.26)} \qquad |h_{n+1}'(x) - h_n'(x)| \le E^{-1}(x)N(x)\frac{[\lambda F(x)]^{n+1}}{(n+1)!},$$

$$n=1,2,\ldots.$$

We see from (A.25) and (A.26) that

$$\sum_{n=0}^{\infty} (h_{n+1}'(x) - h_n'(x))$$

is uniformly convergent on $a \le x < \infty$, so that

$$\text{(A.27)} \qquad \epsilon'(x) = \sum_{n=0}^{\infty} (h_{n+1}'(x) - h_n'(x)).$$

Now from (A.24) we get, on using (A.4) and (A.1),

$$\text{(A.28)} \qquad h_1''(x) = f(x)\,Ai(x) + xh_1(x),$$

while from

$$h_{n+1}'(x)-h_n'(x) = \pi\int_x^\infty [Ai'(x)Bi(t)-Bi'(x)Ai(t)]f(t)$$

$$[h_n(t) - h_{n-1}(t)]\,dt$$

we get, again using (A.4) and (A.1),

$$\text{(A.29)} \qquad h_{n+1}''(x) - h_n''(x) = f(x)\,\{h_n(x) - h_{n-1}(x)\}$$

$$+\ x\{h_{n+1}(x) - h_n(x)\}, \qquad n \geq 1$$

Now the uniform convergence of $\Sigma(h_{n+1}(x)-h_n(x))$ and $\Sigma(h_{n+1}'(x) - h_n'(x))$ implies, by (A.28) and (A.29), the uniform convergence of $\Sigma(h_{n+1}''(x)-h_n''(x))$ and so we have:

$$\text{(A.30)} \qquad \epsilon''(x) = \sum_{n=0}^{\infty} (h_{n+1}''(x) - h_n''(x))$$

$$= f(x)[\epsilon(x) + Ai(x)] + x\epsilon(x).$$

Hence, $w(x) = Ai(x)+\epsilon(x)$ satisfies (A.15). We have already shown that $\epsilon(x)$ satisfies the inequality (A.17). That it satisfies (A.18) follows from (A.5) and the fact that

$$\lim_{x\to\infty} x^{\frac{1}{4}} E(x)\,\epsilon(x) = 0.$$

This last limit relation follows from the following consequence of (A.17):

$$|x^{\frac{1}{4}} E(x)\epsilon(x)| \leq x^{\frac{1}{4}} M(x)(e^{\lambda F(x)}-1),$$

when we use (A.10) and the existence of $\int^{\infty} t^{-\frac{1}{2}}|f(t)|\,dt$.

To see that (A.18) determines $w(x)$ uniquely, we need only notice that the general solution of (A.15) is of the form

$$\text{(A.31)} \qquad u(x) = c_1 w(x) + c_2 w_1(x)$$

where

$$\text{(A.32)} \qquad w_1(x) = w(x) \int_{x_0}^{x} \frac{dt}{w^2(t)}, \text{ for some } x_0.$$

Now, since $w(x) \to 0$, as $x \to \infty$,

it is clear that $\int_{x_0}^{x} \frac{dt}{w^2(t)} \to \infty$ as $x \to \infty$

and hence the condition

$$\lim_{x\to\infty} x^{\frac{1}{4}} E(x)\ u(x) = \tfrac{1}{2}\pi^{-\frac{1}{2}}$$

must imply $c_1 = 1,\ c_2 = 0$.

<u>Proof of Theorem 3.1</u>

The differential equation (3.4),

$$\frac{d^2w}{dt^2} = \{\nu^2 t + q(t)\}w$$

may be transformed to the form (A.15),

$$\frac{d^2w}{dx^2} = \{x + f(x)\}w$$

by taking:

(A.33) $\quad x = \nu^{2/3}t,\ f(x) = \nu^{-4/3}\ q(\nu^{-2/3}\ x).$

On account of (3.3) it is clear that (A.14) is satisfied. Hence a direct application of Theorem A.1 shows that (3.4) possesses a solution

(A.34) $\quad w(\nu,t) = \nu^{2/3}\mathrm{Ai}(\nu^{2/3}t) + \epsilon(\nu,t)$

where

(A.35) $\quad |\epsilon(\nu,t)| \le \nu^{2/3}\ E^{-1}(\nu^{2/3}t)M(\nu^{2/3}t)(\exp[\lambda F_1(t)/\nu]-1).$

Now for $t \geq a$, we have

(A.36) $\quad \exp(\lambda F_1(t)/\nu)-1 \le \exp(\lambda F_1(a)/\nu)-1 = O(\nu^{-1})$

uniformly in t, $a \le t < \infty$, as $\nu\to\infty$.

Since $\dfrac{\mathrm{Ai}(x)\ E(x)}{M\ (x)}$ is bounded away from 0 for $x > 0$ and tends to $1/\sqrt{5}$ as $x \to \infty$ ((A.5) and (A.10)), we

see that there exists a constant c such that

(A.37) $\quad E^{-1}(x)\ M(x) < c\ Ai(x),\ x > 0.$

Substitution of (A.36) and (A.37) in (A.35) gives (3.6).

On the interval $a \le t < \infty$, $E^{-1}(\nu^{2/3}t) \le 1$, $M(\nu^{2/3}t)$ is bounded and, from (A.36),

$$\exp(\lambda F_1(t)/\nu)-1 = O(\nu^{-1}),$$

uniformly in t. Thus (3.7) follows from A.35.

(3.8) follows from (A.35), and the uniqueness assertion follows from the corresponding result in Theorem A.1. This completes the proof of Theorem 3.1.